3.8. Linearization and Differentials :

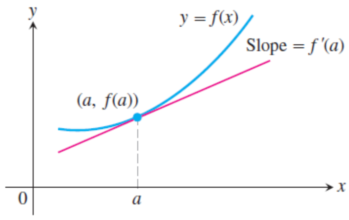

FIGURE The tangent to the curve $y = f(x)$ at $x = a$ is the line $L(x) = f(a) + f'(a)(x - a)$.

In general, the tangent to $y = f(x)$ at a point $x = a$, where f is differentiable (Figure), passes through the point $(a, f(a))$, so its point-slope equation is

$$y = f(a) + f'(a)(x - a).$$

Thus, this tangent line is the graph of the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

For as long as this line remains close to the graph of f, $L(x)$ gives a good approximation to $f(x)$.

DEFINITIONS Linearization, Standard Linear Approximation

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

EXAMPLE Finding a Linearization

Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$ (Figure).

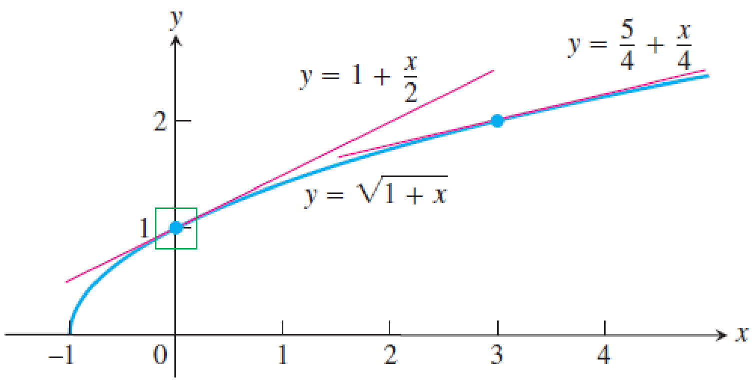

FIGURE The graph of $y = \sqrt{1 + x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.49 shows a magnified view of the small window about 1 on the y-axis.

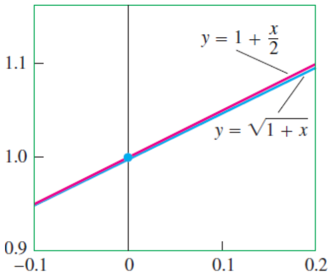

FIGURE Magnified view of the window in Figure

Solution Since

$$f'(x) = \frac{1}{2}(1 + x)^{-1/2},$$

we have $f(0) = 1$ and $f'(0) = 1/2$, giving the linearization

$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

See Figure

Look at how accurate the approximation $\sqrt{1 + x} \approx 1 + (x/2)$ from Example 1 is for values of x near 0.

Differentials

We sometimes use the Leibniz notation dy/dx to represent the derivative of y with respect to x. Contrary to its appearance, it is not a ratio. We now introduce two new variables dx and dy with the property that if their ratio exists, it will be equal to the derivative.

DEFINITION Differential

Let $y = f(x)$ be a differentiable function. The **differential** $\boldsymbol{dx}$ is an independent variable. The **differential** $\boldsymbol{dy}$ is

$$dy = f'(x)\,dx.$$

Unlike the independent variable dx, the variable dy is always a dependent variable. It depends on both x and dx. If dx is given a specific value and x is a particular number in the domain of the function f, then the numerical value of dy is determined.

EXAMPLE Finding the Differential dy

(a) Find dy if $y = x^5 + 37x$.

(b) Find the value of dy when $x = 1$ and $dx = 0.2$.

Solution

(a) $dy = (5x^4 + 37)\,dx$

(b) Substituting $x = 1$ and $dx = 0.2$ in the expression for dy, we have

$$dy = (5 \cdot 1^4 + 37)0.2 = 8.4.$$