# Chapter Four : Application of Derivatives

## Local (Relative) Extreme Values

Figure 4.5 shows a graph with five points where a function has extreme values on its domain $[a, b]$. The function's absolute minimum occurs at $a$ even though at $e$ the function's value is

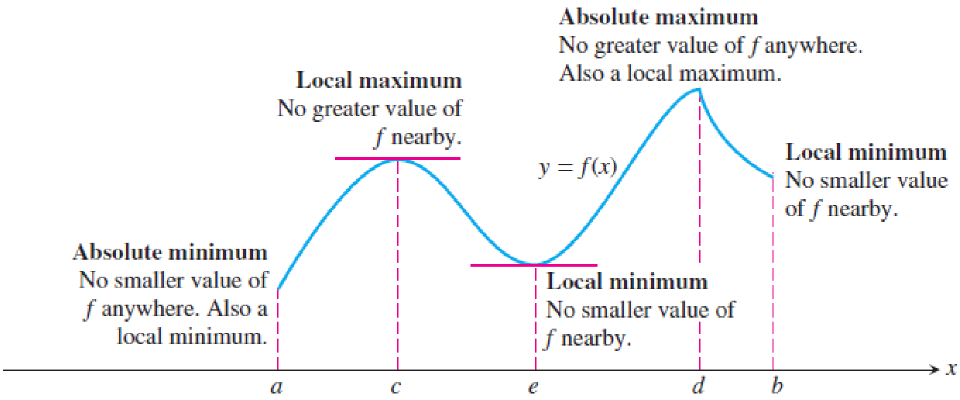


**FIGURE 4.5** How to classify maxima and minima.

smaller than at any other point *nearby*. The curve rises to the left and falls to the right around $c$, making $f(c)$ a maximum locally. The function attains its absolute maximum at $d$.

**DEFINITIONS** **Local Maximum, Local Minimum**

A function $f$ has a **local maximum** value at an interior point $c$ of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function $f$ has a **local minimum** value at an interior point $c$ of its domain if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

## The Mean Value Theorem

The Mean Value Theorem, which was first stated by Joseph-Louis Lagrange, is a slanted version of Rolle's Theorem (Figure 4.14). There is a point where the tangent is parallel to chord $AB$.

**THEOREM 4 The Mean Value Theorem**

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior $(a, b)$. Then there is at least one point $c$ in $(a, b)$ at which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

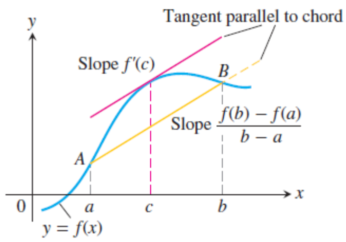


**FIGURE 4.14** Geometrically, the Mean Value Theorem says that somewhere between $A$ and $B$ the curve has at least one tangent parallel to chord $AB$.

**Proof** We picture the graph of $f$ as a curve in the plane and draw a line through the points $A(a, f(a))$ and $B(b, f(b))$ (see Figure 4.15). The line is the graph of the function

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

(point-slope equation). The vertical difference between the graphs of $f$ and $g$ at $x$ is

$$\begin{aligned} h(x) &= f(x) - g(x) \\ &= f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a). \end{aligned}$$

## <u>Curve Sketching with first and second derivatives:</u>

**DEFINITIONS Increasing, Decreasing Function**

Let $f$ be a function defined on an interval $I$ and let $x_1$ and $x_2$ be any two points in $I$.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then $f$ is said to be **increasing** on $I$.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then $f$ is said to be **decreasing** on $I$.

A function that is increasing or decreasing on $I$ is called **monotonic** on $I$.

**COROLLARY First Derivative Test for Monotonic Functions**

Suppose that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

If $f'(x) > 0$ at each point $x \in (a, b)$, then $f$ is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then $f$ is decreasing on $[a, b]$.

### First Derivative Test for Local Extrema

Suppose that $c$ is a critical point of a continuous function $f$, and that $f$ is differentiable at every point in some interval containing $c$ except possibly at $c$ itself. Moving across $c$ from left to right,

1. if $f'$ changes from negative to positive at $c$, then $f$ has a local minimum at $c$;
2. if $f'$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$;
3. if $f'$ does not change sign at $c$ (that is, $f'$ is positive on both sides of $c$ or negative on both sides), then $f$ has no local extremum at $c$.

**EXAMPLE** Using the First Derivative Test for Local Extrema

Find the critical points of

$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}.$$

Identify the intervals on which $f$ is increasing and decreasing. Find the function's local and absolute extreme values.

**Solution** The function $f$ is continuous at all $x$ since it is the product of two continuous functions, $x^{1/3}$ and $(x - 4)$. The first derivative

$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$
$$= \frac{4}{3}x^{-2/3}\left(x - 1\right) = \frac{4(x - 1)}{3x^{2/3}}$$

is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in the domain, so the critical points $x = 0$ and $x = 1$ are the only places where $f$ might have an extreme value.

The critical points partition the $x$-axis into intervals on which $f'$ is either positive or negative. The sign pattern of $f'$ reveals the behavior of $f$ between and at the critical points. We can display the information in a table like the following:

| **Intervals** | $x < 0$ | $0 < x < 1$ | $x > 1$ |
|---|---|---|---|
| **Sign of $f'$** | $-$ | $-$ | $+$ |
| **Behavior of $f$** | decreasing | decreasing | increasing |