

Chapter Five : Integration

Calculus consists of two main branches , the first branch is the differential calculus , and the second branch is the integral calculus.

Integration is the process of calculating an integral.

5.1

Estimating with Finite Sums

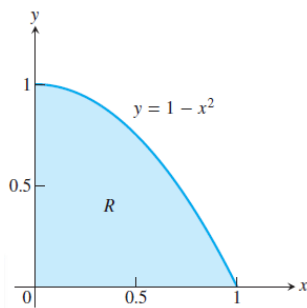


FIGURE 5.1 The area of the region R cannot be found by a simple geometry formula (Example 1).

This section shows how area, average values, and the distance traveled by an object over time can all be approximated by finite sums. Finite sums are the basis for defining the integral in Section 5.3.

Area

The area of a region with a curved boundary can be approximated by summing the areas of a collection of rectangles. Using more rectangles can increase the accuracy of the approximation.

EXAMPLE 1 Approximating Area

What is the area of the shaded region R that lies above the x -axis, below the graph of $y = 1 - x^2$, and between the vertical lines $x = 0$ and $x = 1$? (See Figure 5.1.) An architect might want to know this area to calculate the weight of a custom window with a shape described by R . Unfortunately, there is no simple geometric formula for calculating the areas of shapes having curved boundaries like the region R .

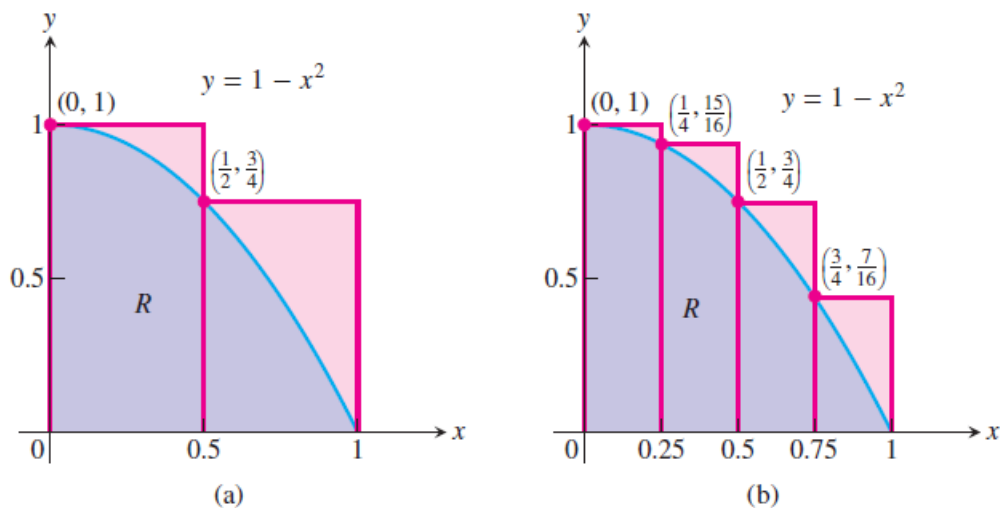


FIGURE 5.2 (a) We get an upper estimate of the area of R by using two rectangles containing R . (b) Four rectangles give a better upper estimate. Both estimates overshoot the true value for the area.

The total area of the two rectangles approximates the area A of the region R ,

$$A \approx 1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{7}{8} = 0.875.$$

Figure 5.2b, we improve our estimate by using four thinner rectangles, each of width $1/4$, which taken together contain the region R . These four rectangles give the approximation

$$A \approx 1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = \frac{25}{32} = 0.78125,$$

Suppose instead we use four rectangles contained *inside* the region R to estimate the area, as in Figure 5.3a. Each rectangle has width $1/4$ as before, but the rectangles are shorter and lie entirely beneath the graph of f . The function $f(x) = 1 - x^2$ is decreasing on $[0, 1]$, so the height of each of these rectangles is given by the value of f at the right endpoint of the subinterval forming its base. The fourth rectangle has zero height and therefore contributes no area. Summing these rectangles with heights equal to the minimum value of $f(x)$ for x a point in each base subinterval, gives a **lower sum** approximation to the area,

$$A \approx \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = \frac{17}{32} = 0.53125.$$

This estimate is smaller than the area A since the rectangles all lie inside of the region R . The true value of A lies somewhere between these lower and upper sums:

$$0.53125 < A < 0.78125.$$

TABLE 5.1 Finite approximations for the area of R

Number of subintervals	Lower sum	Midpoint rule	Upper sum
2	.375	.6875	.875
4	.53125	.671875	.78125
16	.634765625	.6669921875	.697265625
50	.6566	.6667	.6766
100	.66165	.666675	.67165
1000	.6661665	.66666675	.6671665

5.2.: Formula for sums :-

Finite Sums and Sigma Notation

Sigma notation enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

The Greek letter Σ (capital sigma, corresponding to our letter S), stands for “sum.” The **index of summation** k tells us where the sum begins (at the number below the Σ symbol) and where it ends (at the number above Σ). Any letter can be used to denote the index, but the letters i , j , and k are customary.

The index k ends at $k = n$.

The summation symbol
(Greek letter sigma) — $\sum_{k=1}^n a_k$ — a_k is a formula for the k th term.

The index k starts at $k = 1$.

Thus we can write

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2,$$

and

$$f(1) + f(2) + f(3) + \cdots + f(100) = \sum_{i=1}^{100} f(i).$$
