EXAMPLE Applying the Fundamental Theorem

Use the Fundamental Theorem to find

(a) $\dfrac{d}{dx}\displaystyle\int_a^x \cos t\,dt$

(b) $\dfrac{d}{dx}\displaystyle\int_0^x \frac{1}{1+t^2}\,dt$

(c) $\dfrac{dy}{dx}$ if $y = \displaystyle\int_x^5 3t\sin t\,dt$

(d) $\dfrac{dy}{dx}$ if $y = \displaystyle\int_1^{x^2} \cos t\,dt$

Solution

(a) $\dfrac{d}{dx}\displaystyle\int_a^x \cos t\,dt = \cos x$ Eq. 2 with $f(t) = \cos t$

(b) $\dfrac{d}{dx}\displaystyle\int_0^x \frac{1}{1+t^2}\,dt = \frac{1}{1+x^2}$ Eq. 2 with $f(t) = \dfrac{1}{1+t^2}$

(c) Rule 1 for integrals in Table 5.3 of Section 5.3 sets this up for the Fundamental Theorem.

$$\frac{dy}{dx} = \frac{d}{dx}\int_x^5 3t\sin t\,dt = \frac{d}{dx}\left(-\int_5^x 3t\sin t\,dt\right) \qquad \text{Rule 1}$$

$$= -\frac{d}{dx}\int_5^x 3t\sin t\,dt$$

$$= -3x\sin x$$

THEOREM 4 (Continued) The Fundamental Theorem of Calculus Part 2

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

EXAMPLE Evaluating Integrals

(a) $\displaystyle\int_0^{\pi} \cos x\, dx = \sin x \Big]_0^{\pi} = \sin \pi - \sin 0 = 0 - 0 = 0$

(b) $\displaystyle\int_{-\pi/4}^{0} \sec x \tan x\, dx = \sec x \Big]_{-\pi/4}^{0} = \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$

(c) $$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4$$
$$= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right]$$
$$= [8 + 1] - [5] = 4.$$

5.5. Indefinite integrals and the substitution :

If u is any differentiable function, then

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1,\, n \text{ rational}). \tag{1}$$

EXAMPLE Using the Power Rule

$$\int \sqrt{1 + y^2} \cdot 2y\, dy = \int \sqrt{u} \cdot \left(\frac{du}{dy}\right) dy$$
Let $u = 1 + y^2$, $du/dy = 2y$

$$= \int u^{1/2}\, du$$

$$= \frac{u^{(1/2)+1}}{(1/2) + 1} + C$$
Integrate, using Eq. (1) with $n = 1/2$.

$$= \frac{2}{3} u^{3/2} + C$$
Simpler form

$$= \frac{2}{3}(1 + y^2)^{3/2} + C$$
Replace u by $1 + y^2$.

Substitution: Running the Chain Rule Backwards

The substitutions in Examples 1 and 2 are instances of the following general rule.

THEOREM 5 The Substitution Rule
If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Proof The rule is true because, by the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ whenever F is an antiderivative of f:

$$\frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x) \qquad \text{Chain Rule}$$

EXAMPLE Using Substitution

$$\begin{aligned}
\int \cos(7\theta + 5)\,d\theta &= \int \cos u \cdot \frac{1}{7}\,du && \text{Let } u = 7\theta + 5,\ du = 7\,d\theta,\ (1/7)\,du = d\theta. \\
&= \frac{1}{7}\int \cos u\,du && \text{With the } (1/7) \text{ out front, the integral is now in standard form.} \\
&= \frac{1}{7}\sin u + C && \text{Integrate with respect to } u, \text{ Table 4.2.} \\
&= \frac{1}{7}\sin(7\theta + 5) + C && \text{Replace } u \text{ by } 7\theta + 5.
\end{aligned}$$

We can verify this solution by differentiating and checking that we obtain the original function $\cos(7\theta + 5)$. ■