**EXAMPLE 2** Cylindrical Shells Revolving About the $y$-Axis

The region bounded by the curve $y = \sqrt{x}$, the $x$-axis, and the line $x = 4$ is revolved about the $y$-axis to generate a solid. Find the volume of the solid.

**Solution** Sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.21a). Label the segment's height (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.21b, but you need not do that.)

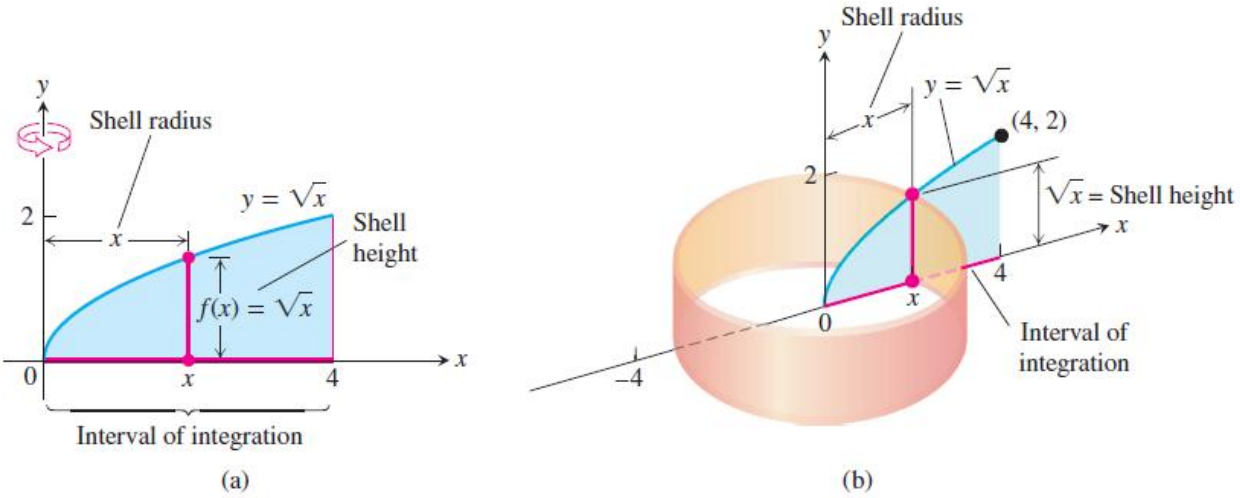


**FIGURE 6.21** (a) The region, shell dimensions, and interval of integration in Example 2. (b) The shell swept out by the vertical segment in part (a) with a width $\Delta x$.

The shell thickness variable is $x$, so the limits of integration for the shell formula are $a = 0$ and $b = 4$ (Figure 6.20). The volume is then

$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx$$

$$= \int_0^4 2\pi(x)\left(\sqrt{x}\right) dx$$

$$= 2\pi \int_0^4 x^{3/2}\, dx = 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}.$$

So far, we have used vertical axes of revolution. For horizontal axes, we replace the $x$'s with $y$'s.

## EXAMPLE 3 Cylindrical Shells Revolving About the $x$-Axis

The region bounded by the curve $y = \sqrt{x}$, the $x$-axis, and the line $x = 4$ is revolved about the $x$-axis to generate a solid. Find the volume of the solid.

**Solution** Sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.22a). Label the segment's length (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.22b, but you need not do that.)

In this case, the shell thickness variable is $y$, so the limits of integration for the shell formula method are $a = 0$ and $b = 2$ (along the $y$-axis in Figure 6.22). The volume of the solid is

$$\begin{aligned} V &= \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy \\ &= \int_0^2 2\pi(y)(4 - y^2)\, dy \\ &= \int_0^2 2\pi(4y - y^3)\, dy \\ &= 2\pi\left[2y^2 - \frac{y^4}{4}\right]_0^2 = 8\pi. \end{aligned}$$

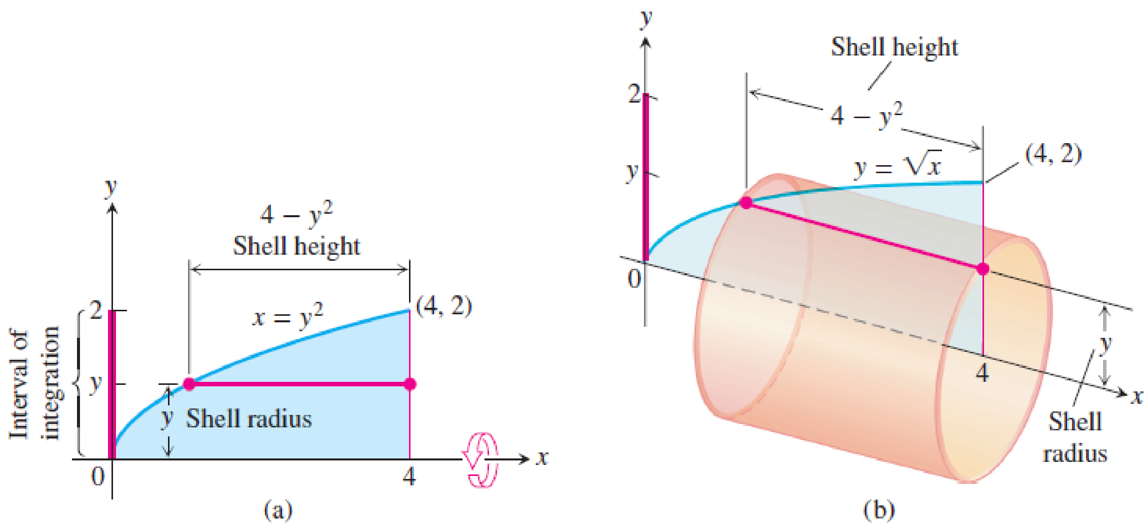


FIGURE 6.22 (a) The region, shell dimensions, and interval of integration in Example 3. (b) The shell swept out by the horizontal segment in part (a) with a width $\Delta y$.

### Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
2. *Find* the limits of integration for the thickness variable.
3. *Integrate* the product $2\pi$ (shell radius) (shell height) with respect to the thickness variable ($x$ or $y$) to find the volume.