EXAMPLE 2 Interpreting the Properties of Logarithms

(a) $\ln 6 = \ln(2 \cdot 3) = \ln 2 + \ln 3$ Product

(b) $\ln 4 - \ln 5 = \ln \frac{4}{5} = \ln 0.8$ Quotient

(c) $\ln \frac{1}{8} = -\ln 8$ Reciprocal

$= -\ln 2^3 = -3 \ln 2$ Power

EXAMPLE 3 Applying the Properties to Function Formulas

(a) $\ln 4 + \ln \sin x = \ln(4 \sin x)$ Product

(b) $\ln \frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$ Quotient

(c) $\ln \sec x = \ln \frac{1}{\cos x} = -\ln \cos x$ Reciprocal

(d) $\ln \sqrt[3]{x+1} = \ln(x+1)^{1/3} = \frac{1}{3}\ln(x+1)$ Power

The Integral $\int (1/u)\, du$

Equation (1) leads to the integral formula

$$\int \frac{1}{u}\, du = \ln u + C \tag{3}$$

when u is a positive differentiable function, but what if u is negative? If u is negative, then $-u$ is positive and

$$\int \frac{1}{u}\, du = \int \frac{1}{(-u)}\, d(-u) \qquad \text{Eq. (3) with } u \text{ replaced by } -u$$

$$= \ln(-u) + C. \tag{4}$$

If u is a differentiable function that is never zero,

$$\int \frac{1}{u}\, du = \ln |u| + C. \tag{5}$$

EXAMPLE 4 Applying Equation (5)

(a) $$\int_0^2 \frac{2x}{x^2-5}\,dx = \int_{-5}^{-1} \frac{du}{u} = \ln|u|\Big]_{-5}^{-1}$$

$u = x^2 - 5,\quad du = 2x\,dx,$
$u(0) = -5,\quad u(2) = -1$

$$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5$$

(b) $$\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3 + 2\sin\theta}\,d\theta = \int_1^5 \frac{2}{u}\,du$$

$u = 3 + 2\sin\theta,\quad du = 2\cos\theta\,d\theta,$
$u(-\pi/2) = 1,\quad u(\pi/2) = 5$

$$= 2\ln|u|\Big]_1^5$$

$$= 2\ln|5| - 2\ln|1| = 2\ln 5$$

Note that $u = 3 + 2\sin\theta$ is always positive on $[-\pi/2, \pi/2]$, so Equation (5) applies. ■

The Integrals of tan *x* and cot *x*

Equation (5) tells us at last how to integrate the tangent and cotangent functions. For the tangent function,

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u}$$

$u = \cos x > 0$ on $(-\pi/2, \pi/2)$,
$du = -\sin x\,dx$

$$= -\int \frac{du}{u} = -\ln|u| + C$$

$$= -\ln|\cos x| + C = \ln\frac{1}{|\cos x|} + C$$ Reciprocal Rule

$$= \ln|\sec x| + C.$$

For the cotangent,

$$\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u}$$

$u = \sin x,$
$du = \cos x\,dx$

$$= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.$$

$$\int \tan u \, du = -\ln|\cos u| + C = \ln|\sec u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C = -\ln|\csc x| + C$$

EXAMPLE 5

$$\int_0^{\pi/6} \tan 2x \, dx = \int_0^{\pi/3} \tan u \cdot \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u \, du$$

$$= \frac{1}{2} \ln|\sec u| \Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

Substitute $u = 2x$, $dx = du/2$, $u(0) = 0$, $u(\pi/6) = \pi/3$

■

Note : Some time we need (ln x) to find the derivative of functions that involve products , quotients, and powers quickly.