EXAMPLE 6 Using Logarithmic Differentiation

Find dy/dx if

$$y = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}, \qquad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

$$\begin{aligned}
\ln y &= \ln \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1} \\
&= \ln ((x^2 + 1)(x + 3)^{1/2}) - \ln (x - 1) && \text{Rule 2} \\
&= \ln (x^2 + 1) + \ln (x + 3)^{1/2} - \ln (x - 1) && \text{Rule 1} \\
&= \ln (x^2 + 1) + \frac{1}{2} \ln (x + 3) - \ln (x - 1). && \text{Rule 3}
\end{aligned}$$

We then take derivatives of both sides with respect to x, using Equation (1) on the left:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 + 1} \cdot 2x + \frac{1}{2} \cdot \frac{1}{x + 3} - \frac{1}{x - 1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

Finally, we substitute for y:

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

■

7.3.The Exponential Function $\boldsymbol{y = e^x}$:

The function $\boldsymbol{y = e^x}$ is the inverse of $y = lnx$.

Then $\quad \boldsymbol{ln(e^x) = x} \quad$, for all x.

$\quad \boldsymbol{e^{lnx} = x} \quad$, for x > 0.

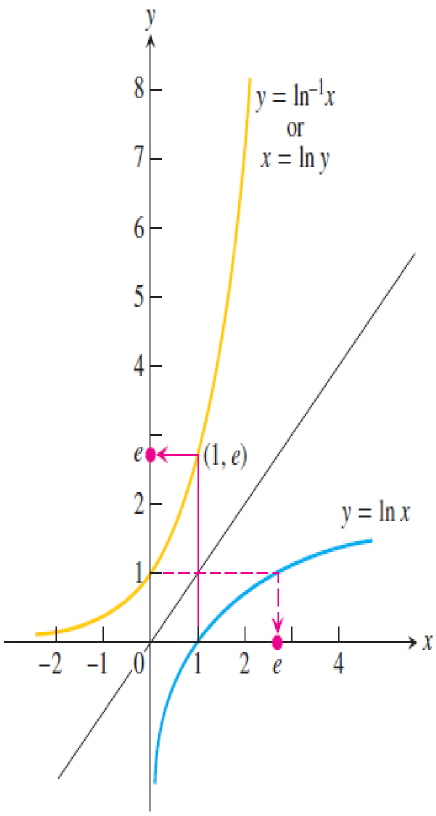

FIGURE 7.11 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$.

The function $\ln x$, being an increasing function of x with domain $(0, \infty)$ and ran $(-\infty, \infty)$, has an inverse $\ln^{-1} x$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. The graph $\ln^{-1} x$ is the graph of $\ln x$ reflected across the line $y = x$. As you can see in Figure 7.11,

$$\lim_{x\to\infty} \ln^{-1} x = \infty \quad \text{and} \quad \lim_{x\to-\infty} \ln^{-1} x = 0.$$

The function $\ln^{-1} x$ is also denoted by $\exp x$.

In Section 7.2 we defined the number e by the equation $\ln(e) = 1$, $e = \ln^{-1}(1) = \exp(1)$. Although e is not a rational number, later in this section we see o way to express it as a limit. In Chapter 11, we will calculate its value with a computer to many places of accuracy as we want with a different formula (Section 11.9, Example To 15 places,

$$e = 2.718281828459045.$$

The Function $y = e^x$

We can raise the number e to a rational power r in the usual way:

$$e^2 = e \cdot e, \qquad e^{-2} = \frac{1}{e^2}, \qquad e^{1/2} = \sqrt{e},$$

and so on. Since e is positive, e^r is positive too. Thus, e^r has a logarithm. When we take t logarithm, we find that

DEFINITION The Natural Exponential Function

For every real number x, $e^x = \ln^{-1} x = \exp x$.