7.4. The Functions $y = a^x$ and $y = \log_a x$:

The Derivative of a^u

We start with the definition $a^x = e^{x \ln a}$:

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) \qquad \frac{d}{dx} e^u = e^u \frac{du}{dx}$$

$$= a^x \ln a .$$

If $a > 0$, then

$$\frac{d}{dx} a^x = a^x \ln a .$$

With the Chain Rule, we get a more general form.

> If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and
>
> $$\frac{d}{dx} a^u = a^u \ln a \, \frac{du}{dx}. \qquad (1)$$

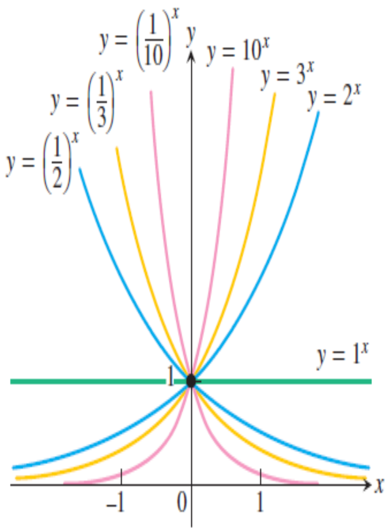

FIGURE 7.12 Exponential functions decrease if $0 < a < 1$ and increase if $a > 1$. As $x \to \infty$, we have $a^x \to 0$ if $0 < a < 1$ and $a^x \to \infty$ if $a > 1$. As $x \to -\infty$, we have $a^x \to \infty$ if $0 < a < 1$ and $a^x \to 0$ if $a > 1$.

EXAMPLE 1 Differentiating General Exponential Functions

(a) $\frac{d}{dx} 3^x = 3^x \ln 3$

(b) $\frac{d}{dx} 3^{-x} = 3^{-x}(\ln 3) \frac{d}{dx}(-x) = -3^{-x} \ln 3$

(c) $\frac{d}{dx} 3^{\sin x} = 3^{\sin x}(\ln 3) \frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3) \cos x$

From Equation (1), we see that the derivative of a^x is positive if $\ln a > 0$, or $a > 1$, and negative if $\ln a < 0$, or $0 < a < 1$. Thus, a^x is an increasing function of x if $a > 1$ and a decreasing function of x if $0 < a < 1$. In each case, a^x is one-to-one. The second derivative

$$\frac{d^2}{dx^2}(a^x) = \frac{d}{dx}(a^x \ln a) = (\ln a)^2 a^x$$

is positive for all x, so the graph of a^x is concave up on every interval of the real line (Figure 7.12).

Other Power Functions

The ability to raise positive numbers to arbitrary real powers makes it possible to define functions like x^x and $x^{\ln x}$ for $x > 0$. We find the derivatives of such functions by rewriting the functions as powers of e.

EXAMPLE 2 Differentiating a General Power Function

Find dy/dx if $y = x^x, \quad x > 0$.

Solution Write x^x as a power of e:

$$y = x^x = e^{x \ln x}. \qquad a^x \text{ with } a = x.$$

Then differentiate as usual:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} e^{x \ln x} \\
&= e^{x \ln x} \frac{d}{dx}(x \ln x) \\
&= x^x \left(x \cdot \frac{1}{x} + \ln x \right) \\
&= x^x (1 + \ln x).
\end{aligned}$$

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The Integral of a^u

If $a \neq 1$, so that $\ln a \neq 0$, we can divide both sides of Equation (1) by $\ln a$ to obtain

$$a^u \frac{du}{dx} = \frac{1}{\ln a} \frac{d}{dx}(a^u).$$

Integrating with respect to x then gives

$$\int a^u \frac{du}{dx}\, dx = \int \frac{1}{\ln a} \frac{d}{dx}(a^u)\, dx = \frac{1}{\ln a} \int \frac{d}{dx}(a^u)\, dx = \frac{1}{\ln a} a^u + C.$$

Writing the first integral in differential form gives

$$\int a^u\, du = \frac{a^u}{\ln a} + C. \qquad (2)$$

EXAMPLE 3 Integrating General Exponential Functions

(a) $\displaystyle\int 2^x\,dx = \frac{2^x}{\ln 2} + C$ Eq. (2) with $a = 2$, $u = x$

(b) $\displaystyle\int 2^{\sin x}\cos x\,dx$

$$= \int 2^u\,du = \frac{2^u}{\ln 2} + C \qquad u = \sin x,\ du = \cos x\,dx, \text{ and Eq. (2)}$$

$$= \frac{2^{\sin x}}{\ln 2} + C \qquad u \text{ replaced by } \sin x$$