

7.8. Hyperbolic Functions:

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x} . The hyperbolic functions simplify many mathematical expressions and they are important in applications. For instance, they are used in problems such as computing the tension in a cable suspended by its two ends, as in an electric transmission line. They also play an important role in finding solutions to differential equations. In this section, we give a brief introduction to hyperbolic functions, their graphs, how their derivatives are calculated, and why they appear as important antiderivatives.

Even and Odd Parts of the Exponential Function

Recall the definitions of even and odd functions from Section 1.4, and the symmetries of their graphs. An even function f satisfies $f(-x) = f(x)$, while an odd function satisfies $f(-x) = -f(x)$. Every function f that is defined on an interval centered at the origin can be written in a unique way as the sum of one even function and one odd function. The decomposition is

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd part}}.$$

If we write e^x this way, we get

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{even part}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{odd part}}.$$

The even and odd parts of e^x , called the hyperbolic cosine and hyperbolic sine of x , respectively, are useful in their own right. They describe the motions of waves in elastic solids and the temperature distributions in metal cooling fins. The centerline of the Gateway Arch to the West in St. Louis is a weighted hyperbolic cosine curve.

Definitions and Identities

The hyperbolic cosine and hyperbolic sine functions are defined by the first two equations in Table 7.5. The table also lists the definitions of the hyperbolic tangent, cotangent, secant, and cosecant. As we will see, the hyperbolic functions bear a number of similarities to the trigonometric functions after which they are named. (See Exercise 84 as well.)

The notation $\cosh x$ is often read “kosh x ,” rhyming with “gosh x ,” and $\sinh x$ is pronounced as if spelled “cinch x ,” rhyming with “pinch x .”

Hyperbolic functions satisfy the identities in Table 7.6. Except for differences in sign, these resemble identities we already know for trigonometric functions.

The second equation is obtained as follows:

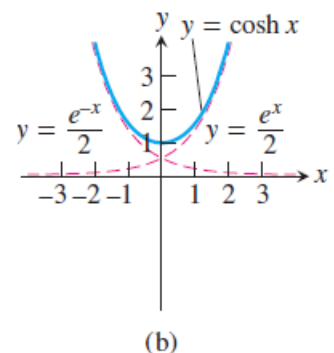
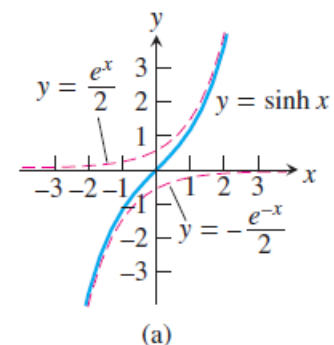
$$\begin{aligned} 2 \sinh x \cosh x &= 2 \left(\frac{e^x - e^{-x}}{2} \right) \left(\frac{e^x + e^{-x}}{2} \right) \\ &= \frac{e^{2x} - e^{-2x}}{2} \\ &= \sinh 2x. \end{aligned}$$

TABLE 7.5 The six basic hyperbolic functions

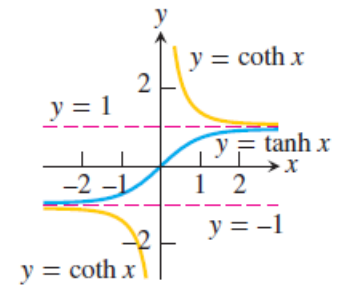
Hyperbolic sine of x : $\sinh x = \frac{e^x - e^{-x}}{2}$

Hyperbolic cosine of x : $\cosh x = \frac{e^x + e^{-x}}{2}$

FIGURE 7.31



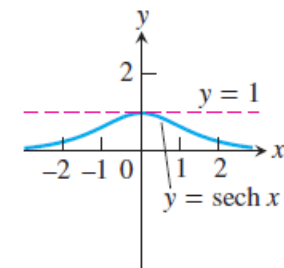
Hyperbolic tangent: $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$



(c)

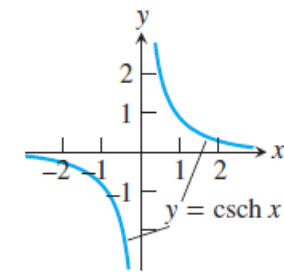
Hyperbolic cotangent: $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Hyperbolic secant: $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$



(d)

Hyperbolic cosecant: $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$



(e)

TABLE 7.6 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$