

Chapter Eight : techniques of Integration

Integration is not as straightforward as differentiation there are no rules that absolutely guarantee obtaining an integral of a function.

In this chapter , we will study techniques for using basic integration formulas to obtain integrals . Also we will study techniques to obtain integrals of more complicated functions.

8.1: Basic Integration Formulas:

In this section , we use techniques to make the integrands in the forms of basic integration formulas.

Table 8.1 shows the basic forms of integrals we have evaluated so far. In this section we present several algebraic or substitution methods to help us use this table. There is a more extensive table at the back of the book; we discuss its use in Section 8.6.

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$	13. $\int \cot u \, du = \ln \sin u + C$ $= -\ln \csc u + C$
2. $\int k \, du = ku + C$ (any number k)	14. $\int e^u \, du = e^u + C$
3. $\int (du + dv) = \int du + \int dv$	15. $\int a^u \, du = \frac{a^u}{\ln a} + C$ ($a > 0, a \neq 1$)
4. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C$ ($n \neq -1$)	16. $\int \sinh u \, du = \cosh u + C$
5. $\int \frac{du}{u} = \ln u + C$	17. $\int \cosh u \, du = \sinh u + C$
6. $\int \sin u \, du = -\cos u + C$	18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \left(\frac{u}{a} \right) + C$
7. $\int \cos u \, du = \sin u + C$	19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \left(\frac{u}{a} \right) + C$
8. $\int \sec^2 u \, du = \tan u + C$	20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \left \frac{u}{a} \right + C$
9. $\int \csc^2 u \, du = -\cot u + C$	21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1} \left(\frac{u}{a} \right) + C$ ($a > 0$)
10. $\int \sec u \tan u \, du = \sec u + C$	22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1} \left(\frac{u}{a} \right) + C$ ($u > a > 0$)
11. $\int \csc u \cot u \, du = -\csc u + C$	
12. $\int \tan u \, du = -\ln \cos u + C$ $= \ln \sec u + C$	

We often have to rewrite an integral to match it to a standard formula.

EXAMPLE 1 Making a Simplifying Substitution

Evaluate

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}} dx.$$

Solution

$$\begin{aligned} \int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}} dx &= \int \frac{du}{\sqrt{u}} && u = x^2 - 9x + 1, \\ &= \int u^{-1/2} du && du = (2x - 9) dx. \\ &= \frac{u^{(-1/2)+1}}{(-1/2) + 1} + C && \text{Table 8.1 Formula 4,} \\ &= 2u^{1/2} + C && \text{with } n = -1/2 \\ &= 2\sqrt{x^2 - 9x + 1} + C \end{aligned}$$

EXAMPLE 2 Completing the Square

Evaluate

$$\int \frac{dx}{\sqrt{8x - x^2}}.$$

Solution We complete the square to simplify the denominator:

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) \\ &= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\ &= \int \frac{du}{\sqrt{a^2 - u^2}} && a = 4, u = (x - 4), \\ &= \sin^{-1} \left(\frac{u}{a} \right) + C && du = dx \\ &= \sin^{-1} \left(\frac{x - 4}{4} \right) + C. && \text{Table 8.1, Formula 18} \end{aligned}$$

EXAMPLE 3 Expanding a Power and Using a Trigonometric Identity

Evaluate

$$\int (\sec x + \tan x)^2 dx.$$

Solution We expand the integrand and get

$$(\sec x + \tan x)^2 = \sec^2 x + 2 \sec x \tan x + \tan^2 x.$$

The first two terms on the right-hand side of this equation are familiar; we can integrate them at once. How about $\tan^2 x$? There is an identity that connects it with $\sec^2 x$:

$$\tan^2 x + 1 = \sec^2 x, \quad \tan^2 x = \sec^2 x - 1.$$

We replace $\tan^2 x$ by $\sec^2 x - 1$ and get

$$\begin{aligned} \int (\sec x + \tan x)^2 dx &= \int (\sec^2 x + 2 \sec x \tan x + \sec^2 x - 1) dx \\ &= 2 \int \sec^2 x dx + 2 \int \sec x \tan x dx - \int 1 dx \\ &= 2 \tan x + 2 \sec x - x + C. \end{aligned}$$

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EXAMPLE 4 Eliminating a Square Root

Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x} dx.$$

Solution We use the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \quad \text{or} \quad 1 + \cos 2\theta = 2 \cos^2 \theta.$$

With $\theta = 2x$, this identity becomes

$$1 + \cos 4x = 2 \cos^2 2x.$$

Hence,

$$\begin{aligned}
 \int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx &= \int_0^{\pi/4} \sqrt{2} \sqrt{\cos^2 2x} \, dx \\
 &= \sqrt{2} \int_0^{\pi/4} |\cos 2x| \, dx && \sqrt{u^2} = |u| \\
 &= \sqrt{2} \int_0^{\pi/4} \cos 2x \, dx && \text{On } [0, \pi/4], \cos 2x \geq 0, \\
 &&& \text{so } |\cos 2x| = \cos 2x. \\
 &= \sqrt{2} \left[\frac{\sin 2x}{2} \right]_0^{\pi/4} && \text{Table 8.1, Formula 7, with} \\
 &&& u = 2x \text{ and } du = 2 \, dx \\
 &= \sqrt{2} \left[\frac{1}{2} - 0 \right] = \frac{\sqrt{2}}{2}.
 \end{aligned}$$

EXAMPLE 5 Reducing an Improper Fraction

Evaluate

$$\int \frac{3x^2 - 7x}{3x + 2} \, dx.$$

Solution The integrand is an improper fraction (degree of numerator greater than or equal to degree of denominator). To integrate it, we divide first, getting a quotient plus a remainder that is a proper fraction:

$$\begin{array}{r}
 \overline{3x + 2 \over 3x^2 - 7x} \\
 \underline{3x^2 + 2x} \\
 -9x \\
 \underline{-9x - 6} \\
 + 6
 \end{array}$$

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2} \, dx = \int \left(x - 3 + \frac{6}{3x + 2} \right) \, dx = \frac{x^2}{2} - 3x + 2 \ln |3x + 2| + C. \quad \blacksquare$$

Reducing an improper fraction by long division (Example 5) does not always lead to an expression we can integrate directly. We see what to do about that in Section 8.5.

EXAMPLE 6 Separating a Fraction

Evaluate

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} dx = 3 \int \frac{x dx}{\sqrt{1 - x^2}} + 2 \int \frac{dx}{\sqrt{1 - x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \quad du = -2x dx, \quad \text{and} \quad x dx = -\frac{1}{2} du.$$

$$\begin{aligned} 3 \int \frac{x dx}{\sqrt{1 - x^2}} &= 3 \int \frac{(-1/2) du}{\sqrt{u}} = -\frac{3}{2} \int u^{-1/2} du \\ &= -\frac{3}{2} \cdot \frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1 - x^2} + C_1 \end{aligned}$$

The second of the new integrals is a standard form,

$$2 \int \frac{dx}{\sqrt{1 - x^2}} = 2 \sin^{-1} x + C_2.$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} dx = -3\sqrt{1 - x^2} + 2 \sin^{-1} x + C. \quad \blacksquare$$

The final example of this section calculates an important integral by the algebraic technique of multiplying the integrand by a form of 1 to change the integrand into one we can integrate.