EXAMPLE 6 Finding Area

Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

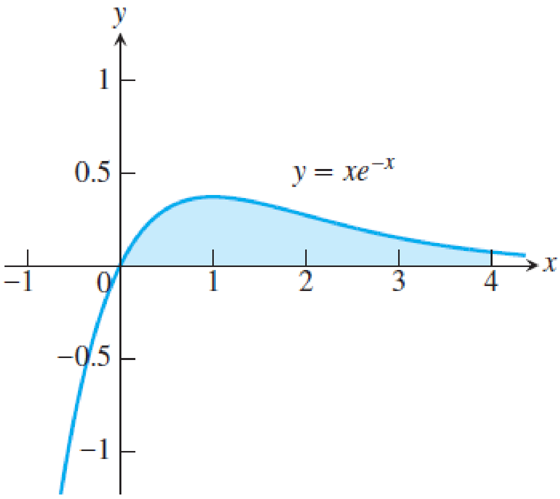

FIGURE 8.1 The region in Example 6.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x}\,dx.$$

Let $u = x$, $dv = e^{-x}\,dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\begin{aligned}\int_0^4 xe^{-x}\,dx &= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\,dx \\ &= [-4e^{-4} - (0)] + \int_0^4 e^{-x}\,dx \\ &= -4e^{-4} - e^{-x}\Big]_0^4 \\ &= -4e^{-4} - e^{-4} - (-e^0) = 1 - 5e^{-4} \approx 0.91.\end{aligned}$$

EXAMPLE 7 Using Tabular Integration

Evaluate

$$\int x^2 e^x \, dx .$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C .$$

EXAMPLE 8 Using Tabular Integration

Evaluate

$$\int x^3 \sin x \, dx .$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C .$$

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Exercises 8.2 :

1. $u = x,\ du = dx;\ dv = \sin\frac{x}{2}\,dx,\ v = -2\cos\frac{x}{2}$;

$$\int x \sin\frac{x}{2}\,dx = -2x\cos\frac{x}{2} - \int\left(-2\cos\frac{x}{2}\right)dx = -2x\cos\left(\frac{x}{2}\right) + 4\sin\left(\frac{x}{2}\right) + C$$

4.

$$\begin{array}{lcl} & & \sin x \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$

$$\int x^2 \sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C$$

5. $u = \ln x,\ du = \frac{dx}{x};\ dv = x\,dx,\ v = \frac{x^2}{2}$;

$$\int_1^2 x\ln x\,dx = \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2 \frac{x^2}{2}\,\frac{dx}{x} = 2\ln 2 - \left[\frac{x^2}{4}\right]_1^2 = 2\ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$$

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7. $u = \tan^{-1} y,\ du = \frac{dy}{1+y^2};\ dv = dy,\ v = y;$

$$\int \tan^{-1} y\, dy = y \tan^{-1} y - \int \frac{y\, dy}{(1+y^2)} = y \tan^{-1} y - \frac{1}{2} \ln(1+y^2) + C = y \tan^{-1} y - \ln\sqrt{1+y^2} + C$$

13.

		e^x
$x^2 - 5x$	$\xrightarrow{(+)}$	e^x
$2x - 5$	$\xrightarrow{(-)}$	e^x
2	$\xrightarrow{(+)}$	e^x
0		

$$\int (x^2 - 5x)\, e^x\, dx = (x^2 - 5x)\, e^x - (2x - 5)e^x + 2e^x + C = x^2 e^x - 7xe^x + 7e^x + C$$
$$= (x^2 - 7x + 7)\, e^x + C$$

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20. $u = \sin^{-1}(x^2),\ du = \frac{2x\, dx}{\sqrt{1-x^4}};\ dv = 2x\, dx,\ v = x^2;$

$$\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2)\, dx = [x^2 \sin^{-1}(x^2)]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2 \cdot \frac{2x\, dx}{\sqrt{1-x^4}} = \left(\tfrac{1}{2}\right)\left(\tfrac{\pi}{6}\right) + \int_0^{1/\sqrt{2}} \frac{d(1-x^4)}{2\sqrt{1-x^4}}$$
$$= \frac{\pi}{12} + \left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}} = \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12}$$

24. $\int e^{-2x} \sin 2x\, dx;\ [y = 2x] \rightarrow \frac{1}{2} \int e^{-y} \sin y\, dy = I;\ [u = \sin y,\ du = \cos y\, dy;\ dv = e^{-y}\, dy,\ v = -e^{-y}]$

$$\Rightarrow I = \frac{1}{2}\left(-e^{-y} \sin y + \int e^{-y} \cos y\, dy\right)\ [u = \cos y,\ du = -\sin y;\ dv = e^{-y}\, dy,\ v = -e^{-y}]$$
$$\Rightarrow I = -\frac{1}{2} e^{-y} \sin y + \frac{1}{2}\left(-e^{-y} \cos y - \int (-e^{-y})(-\sin y)\, dy\right) = -\frac{1}{2} e^{-y}(\sin y + \cos y) - I + C'$$
$$\Rightarrow 2I = -\frac{1}{2} e^{-y}(\sin y + \cos y) + C' \Rightarrow I = -\frac{1}{4} e^{-y}(\sin y + \cos y) + C = -\frac{e^{-2x}}{4}(\sin 2x + \cos 2x) + C, \text{ where}$$
$C = \frac{C'}{2}$