8.3: Integration of Rational Functions by Partial Functions:

Sometimes we need to reverse the rational function to smaller partial fractions to find the integral.

If we have the rational function $\frac{f(x)}{g(x)}$. We can write the function $\frac{f(x)}{g(x)}$ as a sum of partial fraction if :

1) The degree of f(x) is less than the degree of g(x) . If it is not use the long division.

2) we must know the factors of g(x).

We study four cases:

Case 1: Distinct linear factors of g(x):

$$\frac{\boldsymbol{f(x)}}{\boldsymbol{g(x)}} = \frac{\boldsymbol{A}}{\boldsymbol{a_1x+b_1}} + \frac{\boldsymbol{B}}{\boldsymbol{a_2x+b_2}} + \frac{\boldsymbol{C}}{\boldsymbol{a_3x+b_3}}$$

EXAMPLE 1 Distinct Linear Factors

Evaluate

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$$

using partial fractions.

Solution The partial fraction decomposition has the form

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}.$$

To find the values of the undetermined coefficients A, B, and C we clear fractions and get

$$\begin{aligned} x^2+4x+1 &= A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1) \\ &= (A+B+C)x^2 + (4A+2B)x + (3A-3B-C). \end{aligned}$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x obtaining

$$\begin{array}{ll} \text{Coefficient of } x^2: & A+B+C=1 \\ \text{Coefficient of } x^1: & 4A+2B=4 \\ \text{Coefficient of } x^0: & 3A-3B-C=1 \end{array}$$

There are several ways for solving such a system of linear equations for the unknowns A, B, and C, including elimination of variables, or the use of a calculator or computer. Whatever method is used, the solution is $A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\,dx = \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right] dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K,$$

where K is the arbitrary constant of integration (to avoid confusion with the undetermined coefficient we labeled as C). ■

Case 2: Repeated linear factors of g(x):

$$\frac{f(x)}{g(x)} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3}$$

EXAMPLE 2 A Repeated Linear Factor

Evaluate

$$\int \frac{6x+7}{(x+2)^2}\,dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$6x + 7 = A(x+2) + B \qquad \text{Multiply both sides by } (x+2)^2.$$

$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\int \frac{6x+7}{(x+2)^2}\,dx = \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right) dx$$

$$= 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx$$

$$= 6 \ln |x+2| + 5(x+2)^{-1} + C$$

■

EXAMPLE 3 Integrating an Improper Fraction

Evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r@{}l} & 2x \\ x^2 - 2x - 3 \,\big) & \overline{2x^3 - 4x^2 - x - 3} \\ & \underline{2x^3 - 4x^2 - 6x\phantom{{}-3}} \\ & \phantom{2x^3 - 4x^2 -{}} 5x - 3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx = \int 2x\,dx + \int \frac{5x - 3}{x^2 - 2x - 3}\,dx$$

$$= \int 2x\,dx + \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

$$= x^2 + 2\ln|x+1| + 3\ln|x-3| + C.$$

■

Case 3: Distinct irreducible quadratic factors of g(x):

$$\frac{f(x)}{g(x)} = \frac{Ax+B}{a_1x^2+b_1x+c_1} + \frac{Cx+D}{a_2x^2+b_2x+c_2} + \frac{Ex+F}{a_3x^2+b_3x+c_3}$$

EXAMPLE 4 Integrating with an Irreducible Quadratic Factor in the Denominator

Evaluate

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$$

using partial fractions.

Solution The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}. \tag{2}$$

Clearing the equation of fractions gives

$$\begin{aligned} -2x+4 &= (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1) \\ &= (A+C)x^3 + (-2A+B-C+D)x^2 \\ &\quad + (A-2B+C)x + (B-C+D). \end{aligned}$$

Equating coefficients of like terms gives

Coefficients of x^3: $\quad 0 = A + C$

Coefficients of x^2: $\quad 0 = -2A + B - C + D$

Coefficients of x^1: $\quad -2 = A - 2B + C$

Coefficients of x^0: $\quad 4 = B - C + D$

We solve these equations simultaneously to find the values of A, B, C, and D:

$-4 = -2A, \quad A = 2$ — Subtract fourth equation from second.

$C = -A = -2$ — From the first equation

$B = 1$ — $A = 2$ and $C = -2$ in third equation.

$D = 4 - B + C = 1.$ — From the fourth equation

We substitute these values into Equation (2), obtaining

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}.$$

Finally, using the expansion above we can integrate:

$$\begin{aligned}\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\,dx &= \int \left(\frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}\right) dx \\ &= \int \left(\frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}\right) dx \\ &= \ln(x^2 + 1) + \tan^{-1} x - 2\ln|x - 1| - \frac{1}{x - 1} + C. \quad \blacksquare\end{aligned}$$

Case 3: Repeated irreducible quadratic factors of g(x):

$$\frac{f(x)}{g(x)} = \frac{Ax + B}{ax^2 + bx + c} + \frac{Cx + D}{(ax^2 + bx + c)^2} + \frac{Ex + F}{ax^2 + bx + c^3}$$

EXAMPLE 5 A Repeated Irreducible Quadratic Factor

Evaluate

$$\int \frac{dx}{x(x^2 + 1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

Multiplying by $x(x^2 + 1)^2$, we have

$$\begin{aligned}1 &= A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \\ &= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\ &= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A\end{aligned}$$

If we equate coefficients, we get the system

$$A + B = 0, \qquad C = 0, \qquad 2A + B + D = 0, \qquad C + E = 0, \qquad A = 1.$$

Solving this system gives $A = 1$, $B = -1$, $C = 0$, $D = -1$, and $E = 0$. Thus,

$$\begin{aligned}
\int \frac{dx}{x(x^2+1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx \\
&= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2} \\
&= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} \qquad \begin{array}{l} u = x^2+1, \\ du = 2x\,dx \end{array} \\
&= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K \\
&= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K \\
&= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K.
\end{aligned}$$

■

Exercises 8.3 :

1. $\frac{5x-13}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} \Rightarrow 5x - 13 = A(x-2) + B(x-3) = (A+B)x - (2A+3B)$

$\Rightarrow \left.\begin{array}{l} A + B = 5 \\ 2A + 3B = 13 \end{array}\right\} \Rightarrow -B = (10-13) \Rightarrow B = 3 \Rightarrow A = 2$; thus, $\frac{5x-13}{(x-3)(x-2)} = \frac{2}{x-3} + \frac{3}{x-2}$

---

4. $\frac{2x+2}{x^2-2x+1} = \frac{2x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 2x + 2 = A(x-1) + B = Ax + (-A+B) \Rightarrow \left.\begin{array}{l} A = 2 \\ -A + B = 2 \end{array}\right\}$

$\Rightarrow A = 2$ and $B = 4$; thus, $\frac{2x+2}{x^2-2x+1} = \frac{2}{x-1} + \frac{4}{(x-1)^2}$

7. $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{5t+2}{t^2-5t+6}$ (after long division); $\frac{5t+2}{t^2-5t+6} = \frac{5t+2}{(t-3)(t-2)} = \frac{A}{t-3} + \frac{B}{t-2}$

$\Rightarrow 5t + 2 = A(t-2) + B(t-3) = (A+B)t + (-2A-3B) \Rightarrow \left.\begin{array}{l} A + B = 5 \\ -2A - 3B = 2 \end{array}\right\} \Rightarrow -B = (10+2) = 12$

$\Rightarrow B = -12 \Rightarrow A = 17$; thus, $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{17}{t-3} + \frac{-12}{t-2}$

17. $\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}$ (after long division); $\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+2 = A(x+1)+B$

$= Ax + (A+B) \Rightarrow A = 3, A+B = 2 \Rightarrow A = 3, B = -1;\ \int_0^1 \frac{x^3\,dx}{x^2+2x+1}$

$= \int_0^1 (x-2)\,dx + 3\int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2} = \left[\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right]_0^1$

$= \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) = 3\ln 2 - 2$