8.4: Trigonometric Integrals:-

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the work into three cases.

Case 1 If m is odd, we write m as $2k+1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \qquad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x \, dx$, we write n as $2k+1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \qquad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Here are some examples illustrating each case.

EXAMPLE 1 m is Odd

Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

Solution

$$\begin{aligned}\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\ &= \int (1 - \cos^2 x) \cos^2 x \, (-d(\cos x)) \\ &= \int (1 - u^2)(u^2)(-du) && u = \cos x \\ &= \int (u^4 - u^2)\, du \\ &= \frac{u^5}{5} - \frac{u^3}{3} + C \\ &= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.\end{aligned}$$

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EXAMPLE 2 *m* is Even and *n* is Odd

Evaluate

$$\int \cos^5 x \, dx.$$

Solution

$$\begin{aligned}\int \cos^5 x \, dx &= \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 \, d(\sin x) && m = 0 \\ &= \int (1 - u^2)^2 \, du && u = \sin x \\ &= \int (1 - 2u^2 + u^4)\, du \\ &= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.\end{aligned}$$

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EXAMPLE 3 *m* and *n* are Both Even

Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

Solution

$$\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx \\
&= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx \\
&= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx \\
&= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right].
\end{aligned}$$

For the term involving $\cos^2 2x$ we use

$$\begin{aligned}
\int \cos^2 2x \, dx &= \frac{1}{2}\int (1 + \cos 4x)\, dx \\
&= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right).
\end{aligned}$$

Omitting the constant of integration until the final result

For the $\cos^3 2x$ term we have

$$\begin{aligned}
\int \cos^3 2x \, dx &= \int (1 - \sin^2 2x)\cos 2x \, dx \\
&= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right).
\end{aligned}$$

$u = \sin 2x$, $du = 2\cos 2x \, dx$

Again omitting C

Combining everything and simplifying we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

Eliminating Square Roots

In the next example, we use the identity $\cos^2\theta = (1 + \cos 2\theta)/2$ to eliminate a square root.

EXAMPLE 4 Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$$

Solution To eliminate the square root we use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}, \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Therefore,

$$\begin{aligned}
\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx &= \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx \\
&= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx \qquad \begin{array}{l}\cos 2x \geq 0 \\ \text{on } [0, \pi/4]\end{array} \\
&= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1 - 0] = \frac{\sqrt{2}}{2}.
\end{aligned}$$

Integrals of Powers of tan *x* and sec *x*

We know how to integrate the tangent and secant and their squares. To integrate higher powers we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

EXAMPLE 5 Evaluate

$$\int \tan^4 x \, dx.$$

Solution

$$\begin{aligned}
\int \tan^4 x \, dx &= \int \tan^2 x \cdot \tan^2 x \, dx = \int \tan^2 x \cdot (\sec^2 x - 1) \, dx \\
&= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\
&= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx \\
&= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx.
\end{aligned}$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x \, dx$$

and have

$$\int u^2 \, du = \frac{1}{3} u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C.$$

EXAMPLE 6 Evaluate

$$\int \sec^3 x \, dx.$$

Solution We integrate by parts, using

$$u = \sec x, \qquad dv = \sec^2 x \, dx, \qquad v = \tan x, \qquad du = \sec x \tan x \, dx.$$

Then

$$\int \sec^3 x\,dx = \sec x \tan x - \int (\tan x)(\sec x \tan x\,dx)$$

$$= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx \qquad \tan^2 x = \sec^2 x - 1$$

$$= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx.$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

and

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx\,dx, \qquad \int \sin mx \cos nx\,dx, \qquad \text{and} \qquad \int \cos mx \cos nx\,dx$$

arise in many places where trigonometric functions are applied to problems in mathematics and science. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x], \tag{3}$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x], \tag{4}$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]. \tag{5}$$

These come from the angle sum formulas for the sine and cosine functions (Section 1.6). They give functions whose antiderivatives are easily found.

EXAMPLE 7 Evaluate

$$\int \sin 3x \cos 5x\, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$ we get

$$\int \sin 3x \cos 5x\, dx = \frac{1}{2}\int [\sin(-2x) + \sin 8x]\, dx$$

$$= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx$$

$$= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.$$

Exercises 8.4 :

1. $\int_0^{\pi/2} \sin^5 x\, dx = \int_0^{\pi/2} (\sin^2 x)^2 \sin x\, dx = \int_0^{\pi/2} (1-\cos^2 x)^2 \sin x\, dx = \int_0^{\pi/2} (1 - 2\cos^2 x + \cos^4 x)\sin x\, dx$

$= \int_0^{\pi/2} \sin x\, dx - \int_0^{\pi/2} 2\cos^2 x \sin x\, dx + \int_0^{\pi/2} \cos^4 x \sin x\, dx = \left[-\cos x + 2\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5}\right]_0^{\pi/2}$

$= (0) - \left(-1 + \frac{2}{3} - \frac{1}{5}\right) = \frac{8}{15}$

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5. $\int_0^{\pi/2} \sin^7 y\, dy = \int_0^{\pi/2} \sin^6 y \sin y\, dy = \int_0^{\pi/2} (1-\cos^2 y)^3 \sin y\, dy = \int_0^{\pi/2} \sin y\, dy - 3\int_0^{\pi/2} \cos^2 y \sin y\, dy$

$+3\int_0^{\pi/2} \cos^4 y \sin y\, dy - \int_0^{\pi/2} \cos^6 y \sin y\, dy = \left[-\cos y + 3\frac{\cos^3 y}{3} - 3\frac{\cos^5 y}{5} + \frac{\cos^7 y}{7}\right]_0^{\pi/2} = (0) - \left(-1 + 1 - \frac{3}{5} + \frac{1}{7}\right) = \frac{16}{35}$

11. $\int_0^{\pi/2} 35 \sin^4 x \cos^3 x\, dx = \int_0^{\pi/2} 35 \sin^4 x (1 - \sin^2 x)\cos x\, dx = 35\int_0^{\pi/2} \sin^4 x \cos x\, dx - 35\int_0^{\pi/2} \sin^6 x \cos x\, dx$

$= \left[35\frac{\sin^5 x}{5} - 35\frac{\sin^7 x}{7}\right]_0^{\pi/2} = (7-5) - (0) = 2$

12. $\int_0^{\pi} \cos^2 2x \sin 2x\, dx = \left[-\frac{1}{2}\frac{\cos^3 2x}{3}\right]_0^{\pi} = -\frac{1}{6} + \frac{1}{6} = 0$

37. $\int_0^{\pi} \cos 3x \cos 4x \, dx = \frac{1}{2}\int_0^{\pi} (\cos(-x) + \cos 7x) \, dx = \frac{1}{2}\left[-\sin(-x) + \frac{1}{7}\sin 7x\right]_0^{\pi} = \frac{1}{2}(0) = 0$

38. $\int_{-\pi/2}^{\pi/2} \cos 7x \cos x \, dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\cos 6x + \cos 8x) \, dx = \frac{1}{2}\left[\frac{1}{6}\sin 6x + \frac{1}{8}\sin 8x\right]_{-\pi/2}^{\pi/2} = 0$