

## 8.5: Trigonometric Substitutions:

Trigonometric Substitutions enable us to replace the binomials  $(a^2 + u^2)$ ,  $(a^2 - u^2)$ , and  $(u^2 - a^2)$  by single squared terms to find many important integrals.

In this Substitutions:

**For  $(a^2 - u^2)$  use  $u = a \sin \theta$  ,  $-\pi/2 \leq \theta \leq \pi/2$  .**

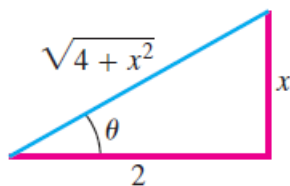
**For  $(a^2 + u^2)$  use  $u = a \tan \theta$  ,  $-\pi/2 < \theta < \pi/2$  .**

**For  $(u^2 - a^2)$  use  $u = a \sec \theta$  ,  $0 \leq \theta \leq \pi$  ,  $\theta \neq \pi/2$**

**EXAMPLE 1** Using the Substitution  $x = a \tan \theta$

Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$



**FIGURE 8.4** Reference triangle for  $x = 2 \tan \theta$  (Example 1):

$$\tan \theta = \frac{x}{2}$$

and

$$\sec \theta = \frac{\sqrt{4 + x^2}}{2}.$$

**Solution** We set

$$x = 2 \tan \theta, \quad dx = 2 \sec^2 \theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4 \tan^2 \theta = 4(1 + \tan^2 \theta) = 4 \sec^2 \theta.$$

Then

$$\begin{aligned} \int \frac{dx}{\sqrt{4 + x^2}} &= \int \frac{2 \sec^2 \theta d\theta}{\sqrt{4 \sec^2 \theta}} = \int \frac{\sec^2 \theta d\theta}{|\sec \theta|} && \sqrt{\sec^2 \theta} = |\sec \theta| \\ &= \int \sec \theta d\theta && \sec \theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{\sqrt{4 + x^2}}{2} + \frac{x}{2} \right| + C && \text{From Fig. 8.4} \\ &= \ln |\sqrt{4 + x^2} + x| + C'. && \text{Taking } C' = C - \ln 2 \end{aligned}$$

Notice how we expressed  $\ln |\sec \theta + \tan \theta|$  in terms of  $x$ : We drew a reference triangle for the original substitution  $x = 2 \tan \theta$  (Figure 8.4) and read the ratios from the triangle. ■

## EXAMPLE 2 Using the Substitution $x = a \sin \theta$

Evaluate

$$\int \frac{x^2 dx}{\sqrt{9 - x^2}}.$$

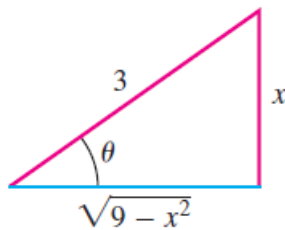
**Solution** We set

$$x = 3 \sin \theta, \quad dx = 3 \cos \theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$9 - x^2 = 9 - 9 \sin^2 \theta = 9(1 - \sin^2 \theta) = 9 \cos^2 \theta.$$

Then

$$\begin{aligned} \int \frac{x^2 dx}{\sqrt{9 - x^2}} &= \int \frac{9 \sin^2 \theta \cdot 3 \cos \theta d\theta}{|3 \cos \theta|} \\ &= 9 \int \sin^2 \theta d\theta && \cos \theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ &= 9 \int \frac{1 - \cos 2\theta}{2} d\theta \\ &= \frac{9}{2} \left( \theta - \frac{\sin 2\theta}{2} \right) + C \\ &= \frac{9}{2} (\theta - \sin \theta \cos \theta) + C && \sin 2\theta = 2 \sin \theta \cos \theta \\ &= \frac{9}{2} \left( \sin^{-1} \frac{x}{3} - \frac{x}{3} \cdot \frac{\sqrt{9 - x^2}}{3} \right) + C && \text{Fig. 8.5} \\ &= \frac{9}{2} \sin^{-1} \frac{x}{3} - \frac{x}{2} \sqrt{9 - x^2} + C. \end{aligned}$$



**FIGURE 8.5** Reference triangle for  $x = 3 \sin \theta$  (Example 2):

$$\sin \theta = \frac{x}{3}$$

and

$$\cos \theta = \frac{\sqrt{9 - x^2}}{3}.$$

**EXAMPLE 3** Using the Substitution  $x = a \sec \theta$ 

Evaluate

$$\int \frac{dx}{\sqrt{25x^2 - 4}}, \quad x > \frac{2}{5}.$$

**Solution** We first rewrite the radical as

$$\begin{aligned}\sqrt{25x^2 - 4} &= \sqrt{25\left(x^2 - \frac{4}{25}\right)} \\ &= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}\end{aligned}$$

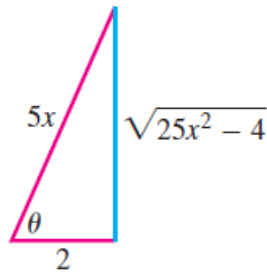
to put the radicand in the form  $x^2 - a^2$ . We then substitute

$$\begin{aligned}x &= \frac{2}{5} \sec \theta, & dx &= \frac{2}{5} \sec \theta \tan \theta d\theta, & 0 < \theta < \frac{\pi}{2} \\ x^2 - \left(\frac{2}{5}\right)^2 &= \frac{4}{25} \sec^2 \theta - \frac{4}{25} \\ &= \frac{4}{25} (\sec^2 \theta - 1) = \frac{4}{25} \tan^2 \theta \\ \sqrt{x^2 - \left(\frac{2}{5}\right)^2} &= \frac{2}{5} |\tan \theta| = \frac{2}{5} \tan \theta. & \text{tan } \theta > 0 \text{ for } 0 < \theta < \pi/2\end{aligned}$$

With these substitutions, we have

$$\begin{aligned}\int \frac{dx}{\sqrt{25x^2 - 4}} &= \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5) \sec \theta \tan \theta d\theta}{5 \cdot (2/5) \tan \theta} \\ &= \frac{1}{5} \int \sec \theta d\theta = \frac{1}{5} \ln |\sec \theta + \tan \theta| + C \\ &= \frac{1}{5} \ln \left| \frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2} \right| + C.\end{aligned}$$

Fig. 8.6



**FIGURE 8.6** If  $x = (2/5)\sec \theta$ ,  $0 < \theta < \pi/2$ , then  $\theta = \sec^{-1}(5x/2)$ , and we can read the values of the other trigonometric functions of  $\theta$  from this right triangle (Example 3).

### **Exercises 8.5 :**

$$1. \ y = 3 \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \ dy = \frac{3 d\theta}{\cos^2 \theta}, \ 9 + y^2 = 9(1 + \tan^2 \theta) = \frac{9}{\cos^2 \theta} \Rightarrow \frac{1}{\sqrt{9+y^2}} = \frac{|\cos \theta|}{3} = \frac{\cos \theta}{3}$$

(because  $\cos \theta > 0$  when  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ );

$$\int \frac{dy}{\sqrt{9+y^2}} = 3 \int \frac{\cos \theta d\theta}{3 \cos^2 \theta} = \int \frac{d\theta}{\cos \theta} = \ln |\sec \theta + \tan \theta| + C' = \ln \left| \frac{\sqrt{9+y^2}}{3} + \frac{y}{3} \right| + C' = \ln |\sqrt{9+y^2} + y| + C$$


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$$3. \ \int_{-2}^2 \frac{dx}{4+x^2} = \left[ \frac{1}{2} \tan^{-1} \frac{x}{2} \right]_{-2}^2 = \frac{1}{2} \tan^{-1} 1 - \frac{1}{2} \tan^{-1} (-1) = \left( \frac{1}{2} \right) \left( \frac{\pi}{4} \right) - \left( \frac{1}{2} \right) \left( -\frac{\pi}{4} \right) = \frac{\pi}{4}$$

$$11. \ y = 7 \sec \theta, 0 < \theta < \frac{\pi}{2}, \ dy = 7 \sec \theta \tan \theta d\theta, \ \sqrt{y^2 - 49} = 7 \tan \theta;$$

$$\int \frac{\sqrt{y^2 - 49}}{y} dy = \int \frac{(7 \tan \theta)(7 \sec \theta \tan \theta) d\theta}{7 \sec \theta} = 7 \int \tan^2 \theta d\theta = 7 \int (\sec^2 \theta - 1) d\theta = 7(\tan \theta - \theta) + C$$

$$= 7 \left[ \frac{\sqrt{y^2 - 49}}{7} - \sec^{-1} \left( \frac{y}{7} \right) \right] + C$$