8.8 Improper Integrals:

DEFINITION Type I Improper Integrals

Integrals with infinite limits of integration are **improper integrals of Type I.**

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

EXAMPLE 1 Evaluating an Improper Integral on $[1, \infty)$

Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is it?

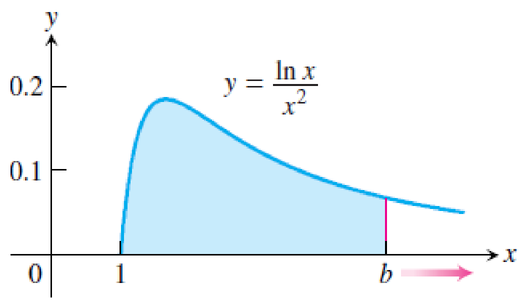

FIGURE 8.19 The area under this curve is an improper integral (Example 1).

Solution We find the area under the curve from $x = 1$ to $x = b$ and examine the limit as $b \to \infty$. If the limit is finite, we take it to be the area under the curve (Figure 8.19). The area from 1 to b is

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

Integration by parts with $u = \ln x$, $dv = dx/x^2$, $du = dx/x$, $v = -1/x$.

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b$$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1.$$

The limit of the area as $b \to \infty$ is

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right]$$

$$= -\left[\lim_{b\to\infty}\frac{\ln b}{b}\right] - 0 + 1$$

$$= -\left[\lim_{b\to\infty}\frac{1/b}{1}\right] + 1 = 0 + 1 = 1.$$

l'Hôpital's Rule

Thus, the improper integral converges and the area has finite value 1. ■

EXAMPLE 2 Evaluating an Integral on $(-\infty, \infty)$

Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}.$$

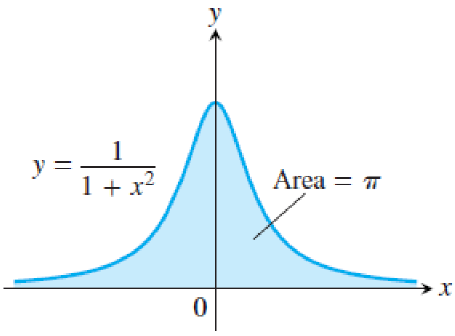

FIGURE 8.20 The area under this curve is finite (Example 2).

Solution According to the definition (Part 3), we can write

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned}
\int_{-\infty}^{0} \frac{dx}{1+x^2} &= \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} \\
&= \lim_{a\to-\infty} \tan^{-1} x \Big]_{a}^{0} \\
&= \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} \\
&= \lim_{b\to\infty} \tan^{-1} x \Big]_{0}^{b} \\
&= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}
\end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Since $1/(1+x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x-axis (Figure 8.20). ■

Integrands with Vertical Asymptotes

Another type of improper integral arises when the integrand has a vertical asymptote—an infinite discontinuity—at a limit of integration or at some point between the limits of integration. If the integrand f is positive over the interval of integration, we can again interpret the improper integral as the area under the graph of f and above the x-axis between the limits of integration.

Consider the region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ (Figure 8.17b). First we find the area of the portion from a to 1 (Figure 8.21).

$$\int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_a^1 = 2 - 2\sqrt{a}$$

Then we find the limit of this area as $a \to 0^+$:

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \left(2 - 2\sqrt{a}\right) = 2.$$

The area under the curve from 0 to 1 is finite and equals

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = 2.$$

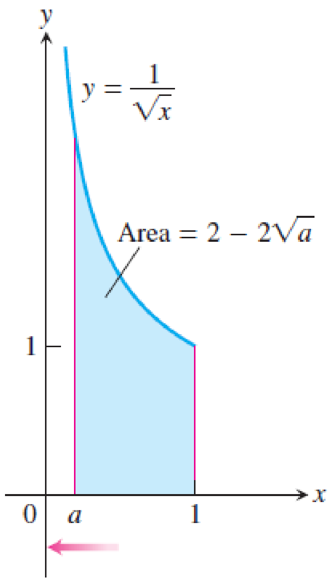

FIGURE 8.21 The area under this curve is

$$\lim_{a \to 0^+} \int_a^1 \left(\frac{1}{\sqrt{x}}\right) dx = 2,$$

an improper integral of the second kind.

DEFINITION Type II Improper Integrals

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

EXAMPLE 4 A Divergent Improper Integral

Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

Solution The integrand $f(x) = 1/(1-x)$ is continuous on $[0, 1)$ but is discontinuous at $x = 1$ and becomes infinite as $x \to 1^-$ (Figure 8.22). We evaluate the integral as

$$\lim_{b\to 1^-}\int_0^b \frac{1}{1-x}\,dx = \lim_{b\to 1^-}\Big[-\ln|1-x|\Big]_0^b$$

$$= \lim_{b\to 1^-}[-\ln(1-b) + 0] = \infty.$$

The limit is infinite, so the integral diverges. ■

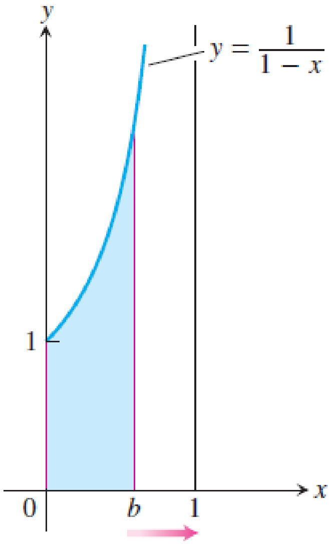

FIGURE 8.22 The limit does not exist:

$$\int_0^1 \left(\frac{1}{1-x}\right)dx = \lim_{b\to 1^-}\int_0^b \frac{1}{1-x}\,dx = \infty.$$

The area beneath the curve and above the x-axis for $[0, 1)$ is not a real number (Example 4).

EXAMPLE 5 Vertical Asympote at an Interior Point

Evaluate

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}.$$

Solution The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0, 1)$ and $(1, 3]$ (Figure 8.23). Thus, by Part 3 of the definition above,

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.$$

Next, we evaluate each improper integral on the right-hand side of this equation.

$$\begin{aligned}\int_0^1 \frac{dx}{(x-1)^{2/3}} &= \lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{b\to 1^-} 3(x-1)^{1/3}\Big]_0^b \\ &= \lim_{b\to 1^-}\left[3(b-1)^{1/3} + 3\right] = 3\end{aligned}$$

$$\begin{aligned}\int_1^3 \frac{dx}{(x-1)^{2/3}} &= \lim_{c\to 1^+}\int_c^3 \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{c\to 1^+} 3(x-1)^{1/3}\Big]_c^3 \\ &= \lim_{c\to 1^+}\left[3(3-1)^{1/3} - 3(c-1)^{1/3}\right] = 3\sqrt[3]{2}\end{aligned}$$

We conclude that

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$

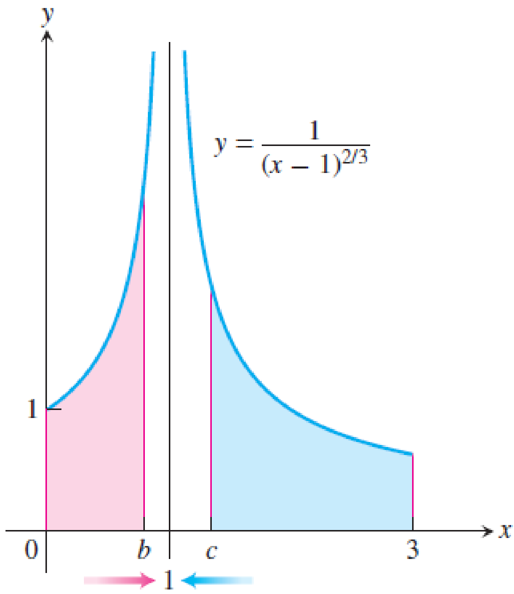

FIGURE 8.23 Example 5 shows the convergence of

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = 3 + 3\sqrt[3]{2},$$

so the area under the curve exists (so it is a real number).