Flow of Compressible Fluid

8.1 Introduction

All fluids are to some degree compressible, compressibility is sufficiently great to affect flow under normal conditions only for a **gas**. If the pressure of the gas does not change by more than about 20%, [or when the change in density more than 5-10 %] it is usually satisfactory to treat the gas as incompressible fluid with a density equal to that at the mean pressure.

When compressibility is taken into account, the equations of flow become more complex than they are for an incompressible fluid.

The flow of gases through orifices, nozzles, and to flow in pipelines presents in all these cases, the flow may reach a limiting maximum value which independent of the downstream pressure (P_2); this is a phenomenon which does not arise with incompressible fluids.

8.2 Velocity of Propagation of a Pressure Wave

The velocity of propagation is a function of *the bulk modulus of elasticity* (ε), where;

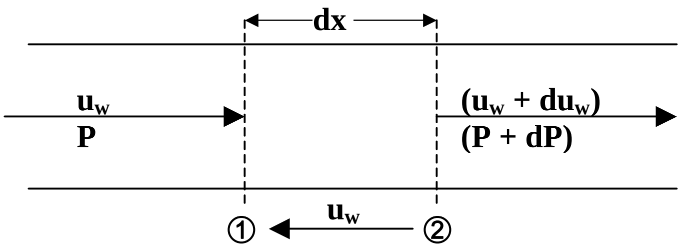
$$\varepsilon = \frac{\textbf{increase of stress within the fluid}}{\textbf{resulting volumetric strain}} = \frac{dP}{-d\upsilon/\upsilon}$$

$$\boxed{\Rightarrow \quad \varepsilon = -\upsilon\frac{dP}{d\upsilon}}$$

where, υ: specific volume ($\upsilon = 1/\rho$).

Suppose a pressure wave to be transmitted at a velocity u_w over a distance dx in a fluid of cross-sectional area A, from section ② to section ① as shown in Figure;

Now imagine the pressure wave to be brought to rest by causing the fluid to flow at a velocity u_w *in the opposite direction*.



From conservation of mass law;

$$\dot{m}_1 = \dot{m}_2$$

$$\rho u_w A = (\rho+d\rho)(u_w + du_w)A$$

$$\Rightarrow \quad \frac{u_w}{\upsilon}A = \frac{(u_w + du_w)}{(\upsilon + d\upsilon)}A$$

and $\dot{m} = \frac{u_w}{\upsilon}A \quad \Rightarrow u_w = \frac{\dot{m}}{A}\upsilon \quad \Rightarrow du_w = \frac{\dot{m}}{A}d\upsilon$

Newton's 2^{nd} law of motion stated that "The rate of change in momentum of fluid is equal to the net force acting on the fluid between sections ① and ②.

Thus;

$$\dot{m}[(u_w - du_w) - u_w] = A[P - (P + dP)] \quad \Rightarrow \frac{\dot{m}}{A}du_w = -dP$$

but $du_w = \frac{\dot{m}}{A}d\upsilon \quad \Rightarrow \frac{\dot{m}}{A}(\frac{\dot{m}}{A}d\upsilon) = -dP \quad \Rightarrow -\frac{dP}{d\upsilon} = \left(\frac{\dot{m}}{A}\right)^2$

we have $-\frac{dP}{d\upsilon} = \frac{\varepsilon}{\upsilon} \quad \Rightarrow \frac{\varepsilon}{\upsilon} = \left(\frac{\dot{m}}{A}\right)^2 = G^2$

$$\Rightarrow \quad \frac{\varepsilon}{\upsilon} = \left(\frac{u_w A/\upsilon}{A}\right)^2 = \left(\frac{u_w}{\upsilon}\right)^2 \quad \Rightarrow \quad u_w^2 = \upsilon\varepsilon$$

$$\boxed{\therefore \quad u_w = \sqrt{\upsilon\varepsilon}}$$

For ideal gases

$P\upsilon^\kappa = const.$ where, κ = 1.0 for isothermal conditions

κ = γ for isentropic conditions, $\gamma = \frac{c_p}{c_v}$

$$d\left(P\upsilon^k\right) = 0 \quad \Rightarrow \quad \upsilon^k dP + Pk\upsilon^{k-1} d\upsilon = 0 \quad \Rightarrow \quad \upsilon^k dP = -kP\frac{\upsilon^k}{\upsilon} d\upsilon \quad \Rightarrow \quad \frac{dP}{d\upsilon} = -k\frac{P}{\upsilon}$$

$$\Rightarrow -\upsilon\left(\frac{dP}{d\upsilon}\right) = kP = \varepsilon \qquad \boxed{\therefore \quad u_w = \sqrt{kP\upsilon}}$$

- For isothermal conditions k = 1 $\boxed{\Rightarrow \quad u_w = \sqrt{P\upsilon}}$
- For isentropic (adiabatic) conditions k = γ $\boxed{\Rightarrow \quad u_w = \sqrt{\gamma P\upsilon}}$

The value of u_W is found to correspond closely to **<u>the velocity of sound in the fluid</u>** and its correspond to the velocity of the fluid at the end of a pipe uder conditions of maximum flow.

<u>Mach Number</u>

Is the ratio between gas velocity to sonic velocity,

$$\boxed{Ma = \frac{u}{u_w}}$$

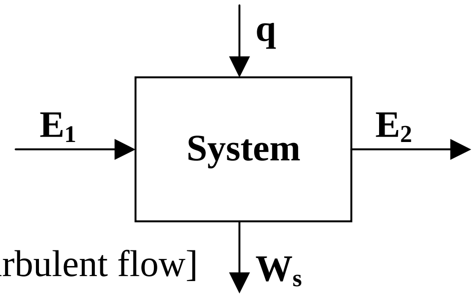
where, Ma > 1 supersonic velocity
Ma = 1 sonic velocity
Ma < 1 subsonic velocity

8.3 <u>General Energy Equation for Compressible Fluids</u>

Let E the total energy per unit mass of the fluid where,

E=Internal energy (U)+Pressure energy (Pυ)+Potential energy(zg)+Kinetic energy ($u^2/2$)

Assume the system in the Figure;



Energy balance

$E_1 + q = E_2 + W_s$

$\Rightarrow E_2 - E_1 = q - W_s$

$\Rightarrow \Delta U + \Delta(P\upsilon) + g\Delta(z) + \Delta(u^2/2) = q - W_s$

[α = 1 for compressible fluid since it almost in turbulent flow]

but $\Delta H = \Delta U + \Delta(P\upsilon)$

$\Rightarrow \Delta H + g\Delta(z) + \Delta(u^2/2) = q - W_s$

$dH + gd(z) + ud(u) = dq - dW_s$

but,

$dH = dq + dF + \upsilon dP$

where,

dF: amount of mechanical energy converted into heat

For irreversible process
$dW = Pd\upsilon - dF$ -------useful work
$dU = dq - dW$ -------closed system
$dH = dU + d(P\upsilon)$
$= dq - dW + d(P\upsilon)$
$= dq - (Pd\upsilon - dF) + d(P\upsilon)$
$= dq - Pd\upsilon + dF + Pd\upsilon + \upsilon dP$
$\Rightarrow dH = dq + dF + \upsilon dP$

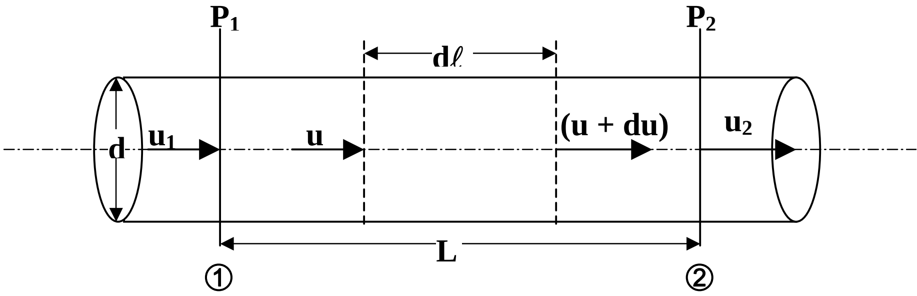
$\Rightarrow$ u du + g dz + υ dP +dW$_s$ + dF = 0

$$\boxed{\therefore \quad \frac{\Delta u^2}{2} + g\Delta z + \int_{P_1}^{P_2} \upsilon dP + Ws + F = 0}$$

General equation of energy apply to any type of fluid

For compressible fluid flowing through (dℓ) of pipe of constant area

u du + g dz + υ dP +dW$_s$ + 4Φ (dℓ/d) u^2 = 0 ----------------------------(*)



$$\dot{m} = \rho u A \quad \Rightarrow \frac{\dot{m}}{A} = \frac{u}{\upsilon} = G$$

$\therefore$ u = G υ $\Rightarrow$ du = G dυ

Substitute these equations into equation (*), to give

G υ (G dυ) + g dz + υ dP +dW$_s$ + 4Φ (dℓ/d) **(**G υ)^2 = 0

For horizontal pipe (dz = 0), and no shaft work (W$_s$ =0)

$\Rightarrow$ G^2 υ (dυ) + υ dP + 4Φ (dℓ/d) **(**G υ)^2 = 0 ----------------------------(**)

Dividing by (υ^2) and integrating over a length L of pipe to give;

$$\boxed{G^2 \ln\left(\frac{\upsilon_2}{\upsilon_1}\right) + \int_{P_1}^{P_2} \frac{dP}{\upsilon} + 4\phi \frac{L}{d} G^2 = 0}$$

Kinetic energy Pressure energy Frictional energy

General equation of energy apply to compressible fluid in horizontal pipe with no shaft work

8.3.1 Isothermal Flow of an Ideal Gas in a Horizontal Pipe

For isothermal conditions of an ideal gas

P υ = constant $\Rightarrow$ P υ = P$_1$ υ$_1$ $\Rightarrow$ 1/υ = P / (P$_1$ υ$_1$)

$$\Rightarrow \int_{P_1}^{P_2} \frac{dP}{\upsilon} = \frac{1}{P_1\upsilon_1}\int_{P_1}^{P_2} P\, dP = \frac{1}{2\, P_1\upsilon_1}\left(P_2^2 - P_1^2\right) \quad \text{-----------------------(1)}$$

P$_1$ υ$_1$ = P$_2$ υ$_2$ $\Rightarrow$ υ$_2$ / υ$_1$ = P$_1$/ P$_2$ -----------------------(2)

Substitute equations (1) and (2) into the genral equation of compressibl fluid to give;

$$\boxed{G^2 \ln\left(\frac{P_1}{P_2}\right) + \frac{(P_2^2 - P_1^2)}{2\, P_1\upsilon_1} + 4\phi \frac{L}{d} G^2 = 0}$$

Let υ$_m$ the mean specific volume at mean pressure P$_m$, where,

P$_m$ =(P$_1$ + P$_2$)/2

P$_m$ υ$_m$ = P$_1$ υ$_1$ $\Rightarrow$ P$_m$ =(P$_1$ + P$_2$)/2 = P$_1$ υ$_1$/ υ$_m$

$$\frac{(P_2^2 - P_1^2)}{2\, P_1\upsilon_1} = \left(\frac{P_2 + P_1}{2}\right)\left(\frac{P_2 - P_1}{P_1\upsilon_1}\right) = \left(\frac{P_1\upsilon_1}{\upsilon_m}\right)\left(\frac{P_2 - P_1}{P_1\upsilon_1}\right)$$

$$\therefore \quad \frac{P_2^2 - P_1^2}{2P_1\upsilon_1} = \frac{P_2 - P_1}{\upsilon_m}$$