Solution by trial and error

υ_2 Assumed	10×10^{-3}	7.5×10^{-3}
υ_2 Calculated	7.5×10^{-3}	7.46×10^{-3}

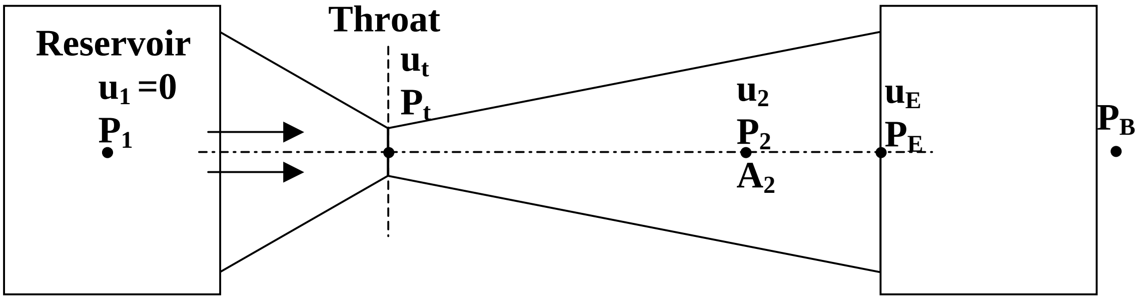
$\Rightarrow \quad \upsilon_2 = 7.46 \times 10^{-3}\ m^3/kg$

$$P_2 = \frac{P_1\upsilon_1}{\upsilon_2} + \frac{\gamma-1}{2\gamma}\frac{G^2}{\upsilon_2}(\upsilon_1^2 - \upsilon_2^2) \Rightarrow \qquad P_2 = 11.94 \times 10^6\ \text{Pa}$$

This value of P_2 in adiabatic conditions is very close to the value in isothermal condition since the actual heat flux is very small.

8.4 <u>Converging-Diverging Nozzles for Gas Flow</u>

Converging-diverging nozzles, sometimes known as “Laval nozzles”, are used for expansion of gases where the pressure drop is large.



P_1: the pressure in the reservoir or initial pressure.
P_2: the pressure at any point in diverging section of the nozzle.
P_E: the pressure at exit of the nozzle.
P_B: the back pressure or the pressure at end.
$P_{critical}$: the pressure at which the velocity of the gas is sonic velocity.

Because the flow rate is large for high-pressure differentials, there is <u>**little time**</u> for heat transfer to take place between the gas and surroundings and the expansion is effectively **isentropic** [adiabatic + reversible]**.**

In these conditions,

$$\frac{\upsilon_2}{\upsilon_1} = \left(\frac{P_1}{P_2}\right)^{\frac{1}{\gamma}} \quad \Rightarrow \upsilon_2 = \upsilon_1\left(\frac{P_2}{P_1}\right)^{-\frac{1}{\gamma}}$$

$$\frac{\Delta u^2}{2} + g\Delta z + \int_{P_1}^{P_2} \upsilon dP + Ws + F = 0$$ the genral energy equation for any type of fluid.

for gas flow from reservoir ($u_1 = 0$) at pressure (P_1) in a horizontal direction, with no shaft work, and by assuming F=0 this equation becomes

$$\frac{u_2^2}{2} + \int_{P_1}^{P_2} \upsilon dP = 0$$ and the pressure energy term is,

$$\int_{P_1}^{P_2} \upsilon dP = \quad \upsilon_1 P_1^{\frac{1}{\gamma}} \int_{P_1}^{P_2} P^{-\frac{1}{\gamma}} dP = \quad \upsilon_1 P_1^{\frac{1}{\gamma}} \left[\frac{P^{1-\frac{1}{\gamma}}}{1-\frac{1}{\gamma}}\right]_{P_1}^{P_2} = \upsilon_1 P_1^{\frac{1}{\gamma}}\left(\frac{\gamma}{\gamma-1}\right)\left[P_2^{\frac{\gamma-1}{\gamma}} - P_1^{\frac{\gamma-1}{\gamma}}\right] \cdots\cdots\cdots \times \frac{P_1^{\frac{\gamma-1}{\gamma}}}{P_1^{\frac{\gamma-1}{\gamma}}}$$

$$\Rightarrow \int_{P_1}^{P_2} \upsilon dP = \left(\frac{\gamma}{\gamma-1}\right) P_1 \upsilon_1 \left[\left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]$$

$$\boxed{\Rightarrow u_2^2 = -2\int_{P_1}^{P_2} \upsilon dP = \left(\frac{2\gamma}{\gamma-1}\right) P_1 \upsilon_1 \left[1 - \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

To estimate the velocity at any point downstream

we have,

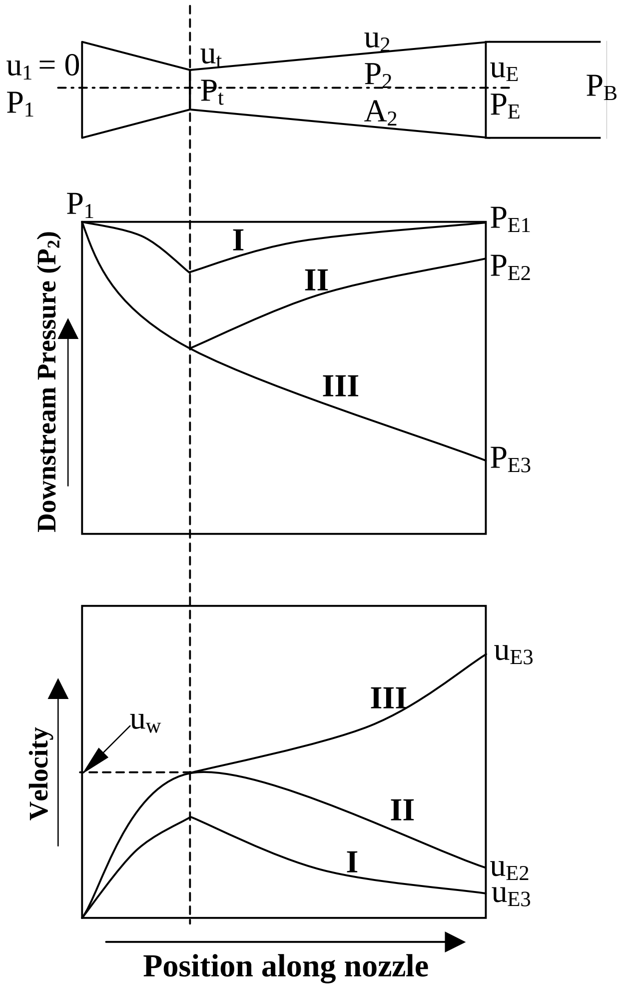
$$\boxed{G_2 = \frac{\dot{m}}{A_2} = \frac{u_2}{\upsilon_2} \quad \Rightarrow A_2 = \dot{m}\frac{\upsilon_2}{u_2}}$$

Cross-sectional area at any point downstream

8.4.1 <u>Maximum Velocity and Critical Pressure Ratio</u>

<u>Critical pressure</u> is the pressure at which the gas reaches <u>sonic velocity</u> [i.e. Ma =1.0].

In converging-diverging nozzles, if the pressure ratio (P_2/P_1) is less than the critical pressure ratio ($P_{critical}/P_1$) (usually, ≈ 0.5) and the velocity at throat is then **equal to** *the velocity of sound*, the effective area for flow presented by nozzle must therefore pass through a minimum. Thus <u>in a converging section</u> the velocity of the gas stream *will never exceed* the sonic velocity, though supersonic velocities *may be obtained* <u>in the diverging section</u> of the converging-diverging nozzle.



<u>Case (I) [P_B high, $P_t > P_{critical}$]</u>

The pressure falls to a minimum at throat [lager than critical pressure] and then rises to a value ($P_{E1}=P_B$). The velocity increase to the maximum at throat [less than sonic velocity] and then decreases to a value of (u_{E1}) at the exit of the nozzle. [**<u>Case (I)</u>** is corresponding to conditions in a venturi meter operating entirely at subsonic velocities]

<u>Case (II) [P_B reduced, $P_B > (P_t = P_{critical})$]</u>

The pressure falls to a critical value at throat where the velocity is sonic. The pressure then rises to a value ($P_{E2}=P_B$) at the exit of the nozzle. The velocity rises to the sonic value at the throat and then falls to a value of (u_{E2}) at the exit of the nozzle.

<u>Case (III) [P_B low, $P_B < (P_t = P_{critical})$]</u>

The pressure falls to a critical value at throat and continues to fall to give an exit pressure ($P_{E3}=P_B$). The velocity rises to the sonic value at the throat and continues to increase to supersonic in the diverging section cone to a value (u_{E3}) at the exit of the nozzle.

With converging-diverging nozzle, <u>the velocity increases beyond the sonic velocity [i.e. reach supersonic velocity]</u> *only if the velocity at the throat is sonic [i.e. critical pressure at throat]* and *the pressure at outlet is lower than the throat pressure*.

8.4.2 The Pressure and Area for Flow

In converging-diverging nozzles, the area required at any point depend upon the ratio of the downstream to upstream pressure (P_2/P_1), and **it is helpful to establish the minimum value of ($A_t = A_2$)**.

$$A_2 = \dot{m}\frac{\upsilon_2}{u_2} \quad \Rightarrow \quad A_2^2 = \dot{m}^2\left(\frac{\upsilon_2}{u_2}\right)^2$$

but $\upsilon_2 = \upsilon_1\left(\frac{P_2}{P_1}\right)^{-\frac{1}{\gamma}}$ and $u_2^2 = \left(\frac{2\gamma}{\gamma-1}\right)P_1\upsilon_1\left[1-\left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}}\right]$

$$\Rightarrow A_2^2 = \dot{m}^2\left(\frac{\gamma-1}{2\gamma}\right)\left[\frac{\upsilon_1^2(P_2/P_1)^{-\frac{2}{\gamma}}}{P_1\upsilon_1\left\{1-(P_2/P_1)^{\frac{\gamma-1}{\gamma}}\right\}}\right] \quad \Rightarrow A_2^2 = \left(\frac{\dot{m}^2\upsilon_1(\gamma-1)}{2\gamma P_1}\right)\left[\frac{(r)^{-\frac{2}{\gamma}}}{\left\{1-(r)^{\frac{\gamma-1}{\gamma}}\right\}}\right]; \quad r = \frac{P_2}{P_1}$$

In the flow stream P_1 falls to P_2 at which minimum A_2 which could be obtain by;

$$\left(\frac{dA_2^2}{dr}\right)_{r=rc} = 0$$

$$\left(\frac{dA_2^2}{dr}\right) = 0 \Rightarrow \left(\frac{\dot{m}^2\upsilon_1(\gamma-1)}{2\gamma P_1}\right)\left[\frac{\left(1-r_c^{\frac{\gamma-1}{\gamma}}\right)\left(\frac{-2}{\gamma}\right)\left(r_c^{-\left(\frac{2+\gamma}{\gamma}\right)}\right)-\left(r_c^{-\frac{2}{\gamma}}\right)\left(-\frac{\gamma-1}{\gamma}r_c^{-\frac{1}{\gamma}}\right)}{\left\{1-(r)^{\frac{\gamma-1}{\gamma}}\right\}^2}\right] = 0$$

$$\Rightarrow \left(1-r_c^{\frac{\gamma-1}{\gamma}}\right)\left(\frac{-2}{\gamma}\right)\left(r_c^{\frac{-2-\gamma}{\gamma}}\right)+\left(r_c^{-\frac{2}{\gamma}}\right)\left(\frac{\gamma-1}{\gamma}\right)\left(r_c^{-\frac{1}{\gamma}}\right) = 0 \quad \Rightarrow \quad \left(\frac{-2}{\gamma}\right)\left(r_c^{\frac{-2-\gamma}{\gamma}}\right)+\left(\frac{\gamma+1}{\gamma}\right)\left(r_c^{-\frac{3}{\gamma}}\right) = 0$$

$$\boxed{\Rightarrow \quad r_c = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}; \qquad \gamma = \frac{c_p}{c_v}; \quad r_c = \frac{P_{critical}}{P_1}}$$ if $\gamma = 1.4 \Rightarrow r_c = 0.528$

$$\therefore A_2^2 = \dot{m}^2\frac{(\gamma-1)}{2\gamma}\left(\frac{\upsilon_1}{P_1}\right)\left[\frac{(P_2/P_1)^{-\frac{2}{\gamma}}}{\left\{1-(P_2/P_1)^{\frac{\gamma-1}{\gamma}}\right\}}\right]$$ The area at any point downstream

and $\Rightarrow \dot{m}^2 = A_2^2\frac{2\gamma}{(\gamma-1)}\frac{P_1}{\upsilon_1}\left(\frac{P_2}{P_1}\right)^{2/\gamma}\left[1-\left(\frac{P_2}{P_1}\right)^{(\gamma-1)/\gamma}\right]$ The mass flow rate

and $\Rightarrow G_2^2 = \frac{\dot{m}^2}{A_2^2}\frac{2\gamma}{(\gamma-1)}\frac{P_1}{\upsilon_1}\left(\frac{P_2}{P_1}\right)^{2/\gamma}\left[1-\left(\frac{P_2}{P_1}\right)^{(\gamma-1)/\gamma}\right]$ The mass velocity

To find the maximum value of (G_2) i.e. $(G_2)_{max}$, set ($dG_2^2/dr = 0$) where, $r = P_2/P_1$ to get the following equation $G_{max} = \sqrt{\gamma P_2/\upsilon_2}$.