(Buckingham's method (or Π-Theorem

It has been observed that the Rayleigh's method of dimensional analysis becomes cumbersome, when a large number of variables are involved. In order to overcome this difficulty, the Buckingham's method may be convenient used. It states that " If there are **(n)** variables in a dimensionally homogeneous equation, and if these variables contain **(m)** fundamental dimensions such as (MLT) they may be grouped into **(n-m)** non-dimensional independent Π-terms".

Mathematically, if a dependent variable X_1 depends upon independent variables (X_2, X_3, X_4, ……… X_n), the functional equation may be written as:

$$X_1 = k\,(X_2, X_3, X_4, \ldots\ldots\ldots X_n)$$

This equation may be written in its general form as;

$$f\,(X_1, X_2, X_3, \ldots\ldots\ldots X_n) = 0$$

In this equation, there are n variables. If there are m fundamental dimensions, then according to Buckingham's Π-theorem;

$$f_1\,(\Pi_1, \Pi_2, \Pi_3, \ldots\ldots\ldots \Pi_{n-m}) = 0$$

The Buckingham's Π-theorem is based on the following steps:

1. First of all, write the functional relationship with the given data.
2. Then write the equation in its general form.
3. Now choose **m** *repeating variables (or recurring set)* and write separate expressions for each Π-term. Every Π-term will contain the repeating variables and one of the remaining variables. **Just** the repeating variables are written in exponential form.
4. With help of the principle of dimensional homogeneity find out the values of powers a, b, c, …… by obtaining simultaneous equations.
5. Now substitute the values of these exponents in the Π-terms.
6. After the Π-terms are determined, write the functional relation in the required form.

Note:-

Any Π-term may be replaced by any power of it, because the power of a non-dimensional term is also non-dimensional e.g. Π_1 may be replaced by Π_1^2, Π_1^3, $\Pi_1^{0.5}$,……. or by $2\Pi_1$, $3\Pi_1$, $\Pi_1/2$, ……etc.

Selection of repeating variables

In the previous section, we have mentioned that we should choose **(m)** repeating variables and write separate expressions for each Π-term. Though there is no hard or fast rule for the selection of repeating variables, yet the following points should be borne in mind while selecting the repeating variables:

1. The variables should be such that none of them is dimension les.
2. No two variables should have the same dimensions.
3. Independent variables should, as far as possible, be selected as repeating variables.
4. Each of the fundamental dimensions must appear in at least one of the **m** variables.
5. It must not possible to form a dimensionless group from some or all the variables within the repeating variables. If it were so possible, this dimensionless group would, of course, be one of the Π-term.
6. In general the selected repeating variables should be expressed as the following: **(1)** representing the flow characteristics, **(2)**, representing the geometry and **(3)** representing the physical properties of fluid.
7. In case of that the example is held up, then one of the repeating variables should be changed.

Example -2

By dimensional analysis, obtain an expression for the drag force (F) on a partially submerged body moving with a relative velocity (u) in a fluid; the other variables being the linear dimension (L), surface roughness (e), fluid density (ρ), and gravitational acceleration (g).

Solution:

Drag force (F) N	$\equiv [MLT^{-2}]$
Relative velocity (u) m/s	$\equiv [LT^{-1}]$
Linear dimension (L) m	$\equiv [L]$
Surface roughness (e) m	$\equiv [L]$
Density (ρ) kg/m^3	$\equiv [ML^{-3}]$
Acceleration of gravity (g) m/s^2	$\equiv [ML^{-1}\ T^{-1}]$

$$F = k\,(u, L, e, \rho, g)$$

$$f\,(F, u, L, e, \rho, g) = 0$$

$$n = 6,\ m = 3,\ \Rightarrow \Pi = n - m = 6 - 3 = 3$$

No. of repeating variables = m = 3

The selected repeating variables is (u, L, ρ)

$$\Pi_1 = u^{a1}\,L^{b1}\,\rho^{c1}\,F \quad \text{--------------(1)}$$

$$\Pi_2 = u^{a2}\,L^{b2}\,\rho^{c2}\,e \quad \text{--------------(2)}$$

$$\Pi_3 = u^{a3}\,L^{b3}\,\rho^{c3}\,g \quad \text{--------------(3)}$$

For Π_1 equation (1)

$$[M^0\,L^0\,T^0] = [L\,T^{-1}]^{a1}\,[L]^{b1}[ML^{-3}]^{c1}[MLT^{-2}]$$

Now applied dimensional homogeneity

For M	$0 = c1 + 1$	$\Rightarrow$	$c1 = -1$
For T	$0 = -a1 - 2$	$\Rightarrow$	$a1 = -2$
For L	$0 = a1 + b1 - 3c1 + 1$	$\Rightarrow$	$b1 = -2$

$\Pi_1 = u^{-2}\, L^{-2}\, \rho^{-1}\, F \qquad \Rightarrow \quad \Pi_1 = \dfrac{F}{u^2\, L^2\, \rho}$

For Π_2 equation (2)

$$[M^0\, L^0\, T^0] = [L\, T^{-1}]^{a2}\, [L]^{b2} [ML^{-3}]^{c2} [L]$$

For M	$0 = c2$	$\Rightarrow$	$c2 = 0$
For T	$0 = -a2$	$\Rightarrow$	$a2 = 0$
For L	$0 = a2 + b2 - 3c2 + 1$	$\Rightarrow$	$b2 = -1$

$\Pi_2 = L^{-1}\, e \qquad \Rightarrow \quad \Pi_2 = \dfrac{e}{L}$

For Π_3 equation (3)

$$[M^0\, L^0\, T^0] = [L\, T^{-1}]^{a3}\, [L]^{b3} [ML^{-3}]^{c3} [L\, T^{-2}]$$

For M	$0 = c3$	$\Rightarrow$	$c3 = 0$
For T	$0 = -a3 - 2$	$\Rightarrow$	$a3 = -2$
For L	$0 = a3 + b3 - 3c3 + 1$	$\Rightarrow$	$b3 = 1$

$\Pi_3 = u^{-2}\, L\, g \qquad \Rightarrow \quad \Pi_3 = \dfrac{L\, g}{u^2}$

$f_1\,(\Pi_1, \Pi_2, \Pi_3) = 0 \qquad \Rightarrow \quad f_1\!\left(\dfrac{F}{u^2\, L^2\, \rho}, \dfrac{e}{L}, \dfrac{L\, g}{u^2}\right) = 0$

$$\therefore\; F = u^2 L^2\, \rho\, f\!\left(\frac{e}{L}, \frac{L\, g}{u^2}\right)$$

Dimensions of some important variables

Item	Property	Symbol	SI Units	M.L.T.
1-	Velocity	u	m/s	LT^{-1}
2-	Angular velocity	ω	Rad/s, Deg/s	T^{-1}
3-	Rotational velocity	N	Rev/s	T^{-1}
4-	Acceleration	a, g	m/s^2	LT^{-2}
5-	Angular acceleration	α	s^{-2}	T^{-2}
6-	Volumetric flow rate	Q	m^3/s	L^3T^{-1}
7-	Discharge	Q	m^3/s	L^3T^{-1}
8-	Mass flow rate	$\dot{m}$	kg/s	MT^{-1}
9-	Mass (flux) velocity	G	$kg/m^2.s$	$ML^{-2}T^{-1}$
10-	Density	ρ	kg/m^3	ML^{-3}
11-	Specific volume	υ	m^3/kg	L^3M
12-	Specific weight	sp.wt	N/m^3	$ML^{-2}T^{-2}$
13-	Specific gravity	sp.gr	[-]	[-]
14-	Dynamic viscosity	μ	kg/m.s, Pa.s	$ML^{-1}T^{-1}$
15-	Kinematic viscosity	ν	m^2/s	L^2T^{-1}
16-	Force	F	N	MLT^{-2}
17-	Pressure	P	$N/m^2 \equiv Pa$	$ML^{-1}T^{-2}$
18-	Pressure gradient	**$\Delta P/L$**	**Pa/m**	$ML^{-2}T^{-2}$
19-	Shear stress	τ	N/m^2	$ML^{-1}T^{-2}$
20-	Shear rate	$\dot{\gamma}$	s^{-1}	T^{-1}
21-	Momentum	M	kg.m/s	MLT^{-1}
22-	Work	W	N.m≡J	ML^2T^{-2}
23-	Moment	M	N.m≡J	ML^2T^{-2}
24-	Torque	Γ	N.m≡J	ML^2T^{-2}
25-	Energy	E	J	ML^2T^{-2}
26-	Power	P	J/s≡W	ML^2T^{-3}
27-	Surface tension	σ	N/m	MT^{-2}
28-	Efficiency	η	[-]	[-]
29-	Head	h	m	L
30-	Modulus of elasticity	ε, K	Pa	$ML^{-1}T^{-2}$

English Units
$g = 32.741\ ft/s^2$
$g_c = 32.741\ lb_m.ft/lb_f.s^2$

SI Units
$g = 9.81\ m/s^2$
$g_c = 1.0\ kg.m/N.s^2$

$psi \equiv lb_f/in^2$ $Pa \equiv Pascal = N/m^2$ $bar = 10^5\ Pa$

$1.0\ atm = 1.01325\ bar = 1.01325*10^5\ Pa = 101.325\ kPa = 14.7\ psi = 760\ torr\ (mmHg)$
$\approx 1.0\ kg/cm^2$

$R = 8.314\ (Pa.m^3/mol.K)$ or $(J/mol.K) = 82.06\ (atm.cm^3/mol.K) = 10.73\ (psi.ft^3/lbmol.R)$
$= 1.987\ (cal/mol.K) = 1.986\ (Btu/lbmol.R) = 1545\ (lb_f.ft/lbmol.R)$