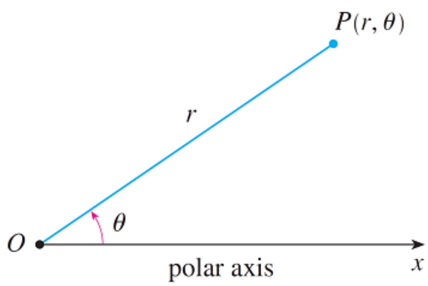
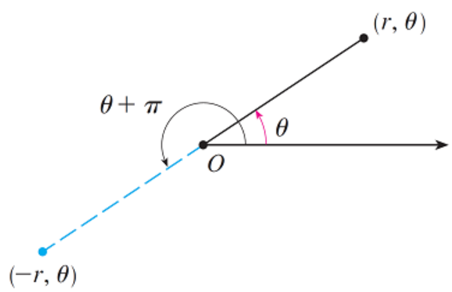
Polar Coordinates

DEFINITION: The polar coordinate system is a two-dimensional coordinate system in which each Point P on a plane is determined by a distance r from a fixed point O that is called the pole (or origin) and an angle ϴ from a fixed direction. The point P is represented by the ordered pair (r , ϴ) and r , ϴ are called polar coordinates.

We extend the meaning of polar coordinates (r , ϴ) to the case in which r is negative by agreeing that the points (- r , ϴ) and (r , ϴ) lie in the same line through O and at the same distance |r | from O but on opposite sides of O: If r >0 , the point (r , ϴ) lies in the same quadrant as ϴ , if r <0; it lies in the quadrant on the opposite side of the pole.

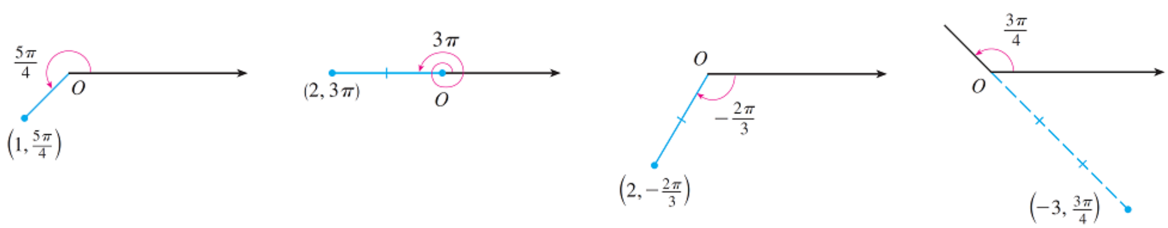




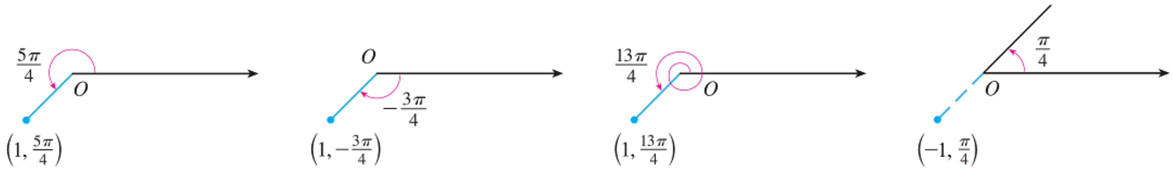
EXAMPLE: Plot the points whose polar coordinates are given:

(a) (1,5π/4) (b) (2,3π) (c) (2, 2π/3) (d) (3,3π/4)

Solution:

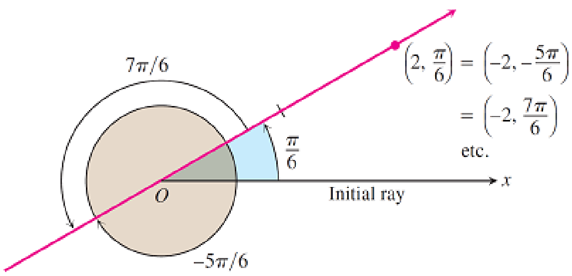


In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point (1;5π/4) in the Example above could be written as (1; 3π/4) or (1;13π/4) or (1; π/4):



EXAMPLE: Find all the polar coordinates of the point P(2; π/6):

Solution: We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of π/6 radians with the initial ray, and mark the point (2; π/6): We then find the angles for the other coordinate pairs of P in which r= 2 and r= -2:



For r= 2; the complete list of angles are

$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \quad \ldots$$

For r= -2; the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \quad \ldots$$

The corresponding coordinate pairs of P are

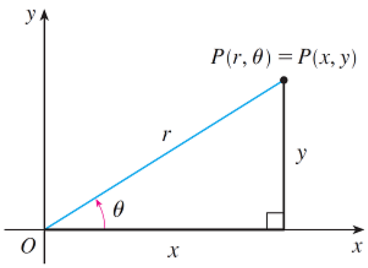
$$\left(2, \frac{\pi}{6} + 2n\pi\right), \quad n = 0, \pm 1, \pm 2, \ldots$$

and

$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \quad n = 0, \pm 1, \pm 2, \ldots$$

When n= 0 ; the formulas give (2, π/6) and (-2, -5π/6):When n= 1;they give (2,13 π/6) and (-2,7 π/6); and so on.

The connection between polar and Cartesian coordinates can be seen from the figure below and described by the following formulas:



$$\boxed{x = r\cos\theta \qquad y = r\sin\theta}$$

$$\boxed{r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}}$$

EXAMPLE:

(a) Convert the point (2; π/3) from polar to Cartesian coordinates.

(b) Represent the point with Cartesian coordinates (1;- 1) in terms of polar coordinates.

Solution:

(a) We have:

(b)

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1 \qquad y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore, the point is (1;√3) in Cartesian coordinates.

(b)If we choose r to be positive, then

$$r = \sqrt{x^2+y^2} = \sqrt{1^2+(-1)^2} = \sqrt{2} \qquad \tan\theta = \frac{y}{x} = -1$$

Since the point $(1,-1)$ lies in the fourth quadrant, we can choose $\theta = -\pi/4$ or $\theta = 7\pi/4$. Thus one possible answer is $(\sqrt{2}, -\pi/4)$; another is $(\sqrt{2}, 7\pi/4)$.

EXAMPLE: Express the equation x= 1 in polar coordinates.

Solution: We use the formula x=r cosΘ:

$$x = 1$$

$$r\cos\theta = 1$$

$$r = \sec\theta$$

EXAMPLE: Express the equation x^2= 4y in polar coordinates.

Solution: We use the formulas x=rcosΘ and y=rsinΘ:

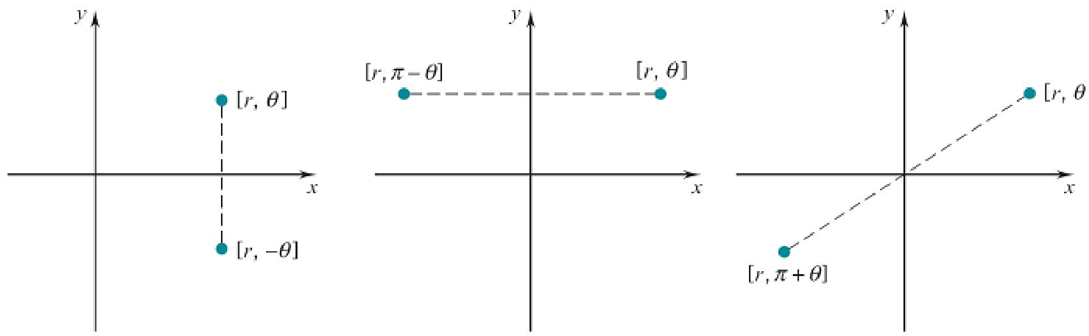
$$x^2 = 4y$$

$$(r\cos\theta)^2 = 4r\sin\theta$$

$$r^2\cos^2\theta = 4r\sin\theta$$

$$r = 4\frac{\sin\theta}{\cos^2\theta} = 4\sec\theta\tan\theta$$

Symmetry



symmetry about the x-axis symmetry about the y-axis symmetry about the origin

Polar Curves

The graph of a polar equation r=f(ϴ); or more generally F(r; ϴ) = 0; consists of all points P that have at least one polar representation (r; ϴ) whose coordinates satisfy the equation.

EXAMPLE: Sketch the polar curve ϴ = 1:

Solution: This curve consists of all points (r; ϴ) such that the polar angle ϴ is 1 radian. It is the straight line that passes through O and makes an angle of 1 radian with the polar axis. Notice that the points (r;1) on the line with r > 0 are in the first quadrant, whereas those with r < 0 are in the third quadrant.