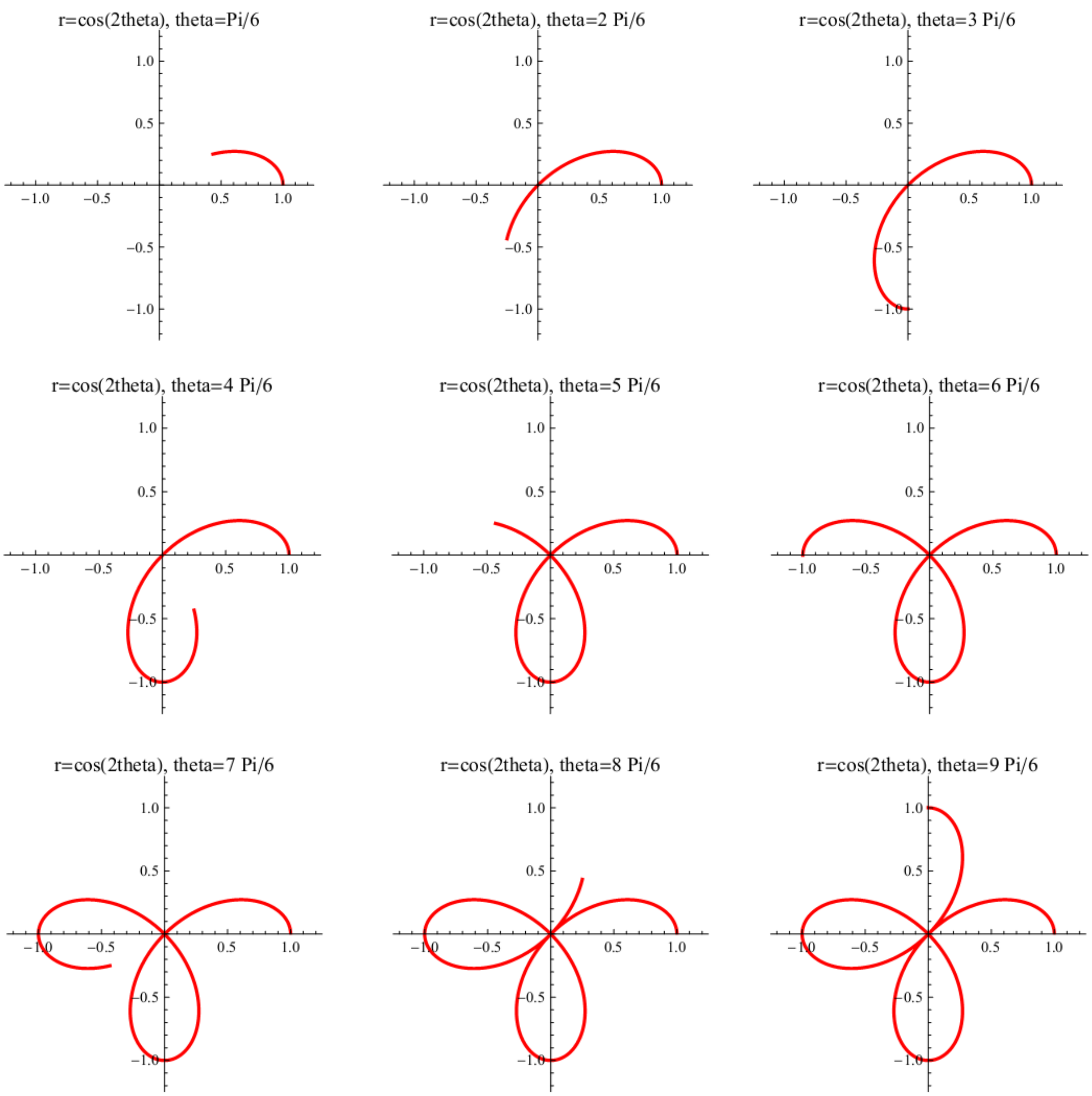
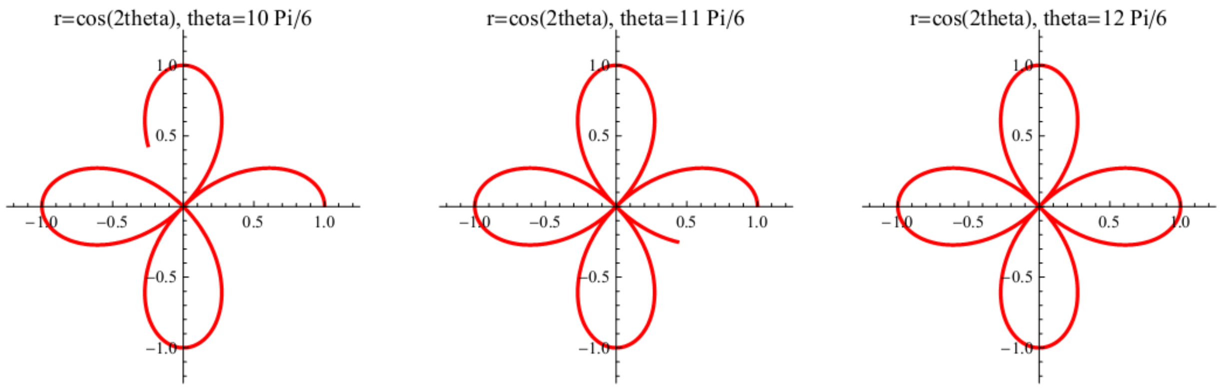
EXAMPLE: Sketch the curve r= cos(2Θ); 0 ≤ Θ ≤ 2π (four-leaved rose).

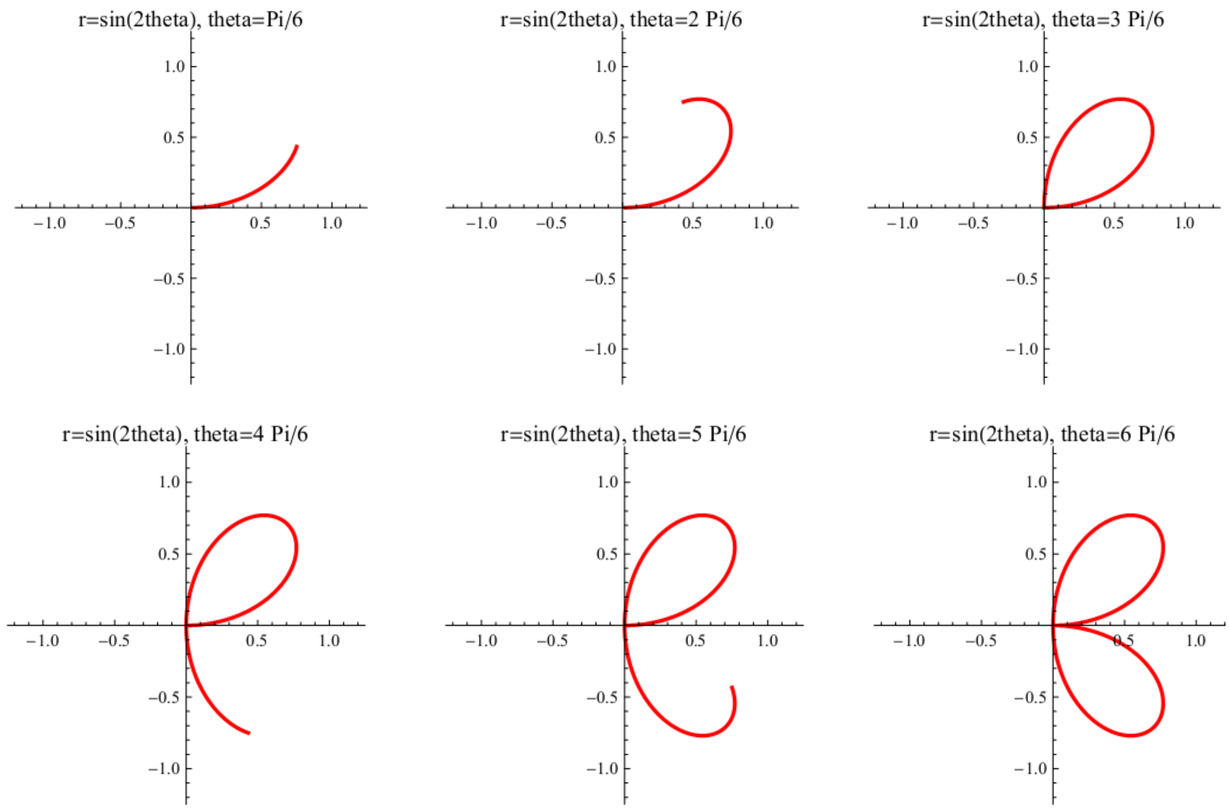
Solution: We have





EXAMPLE: Sketch the curve r= sin(2Θ); 0 ≤ Θ ≤ 2π(four-leaved rose).

Solution: We have



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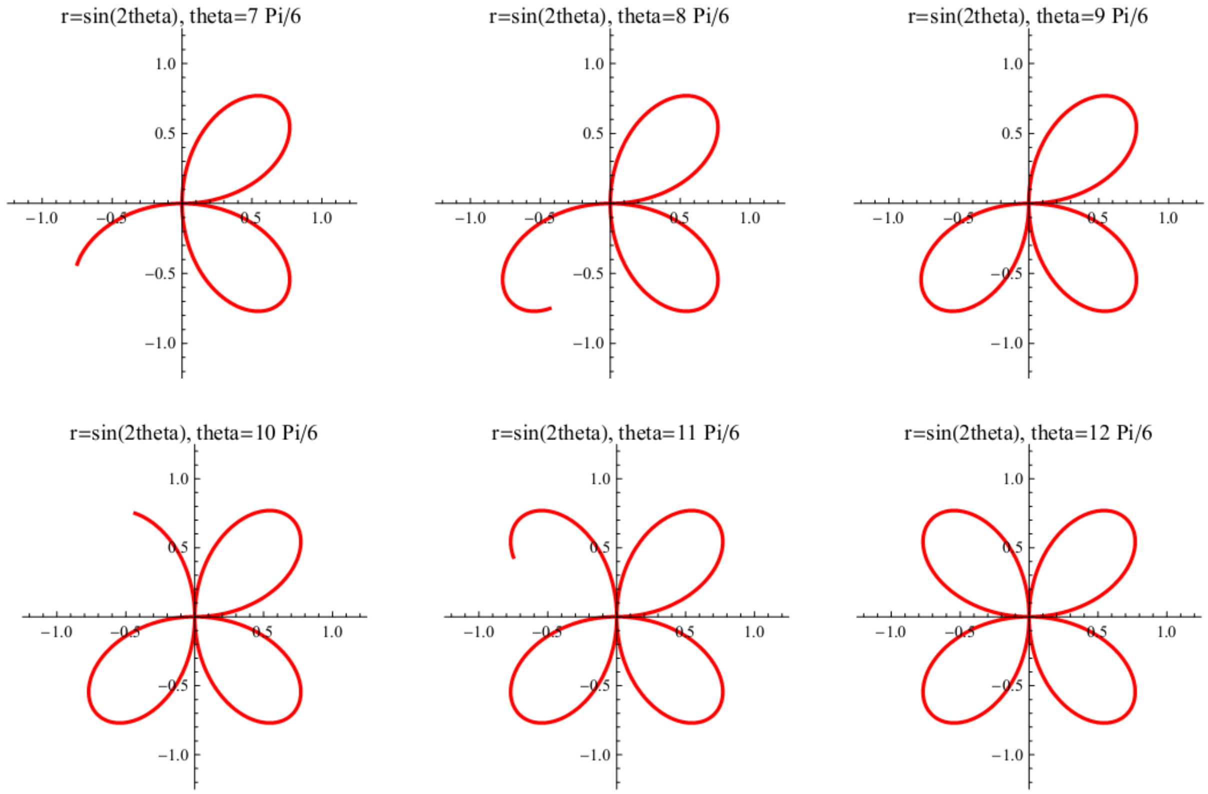
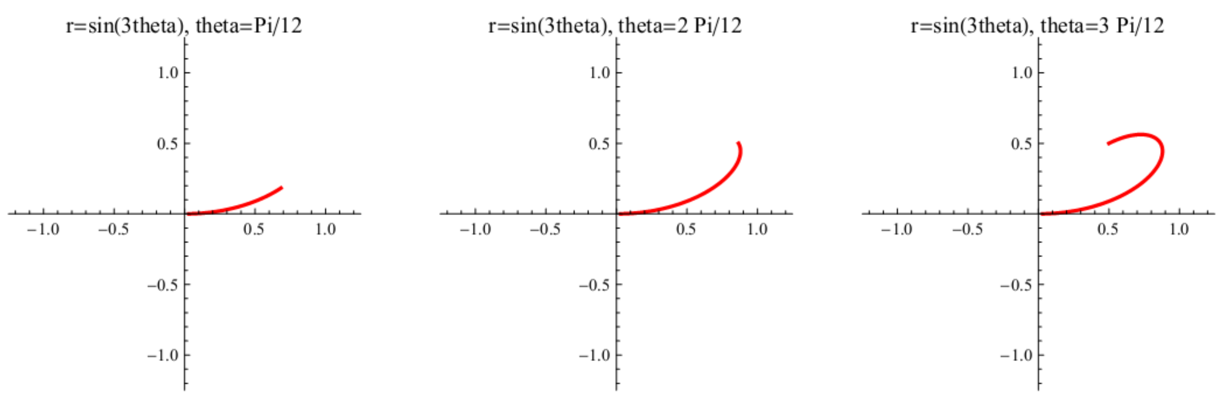
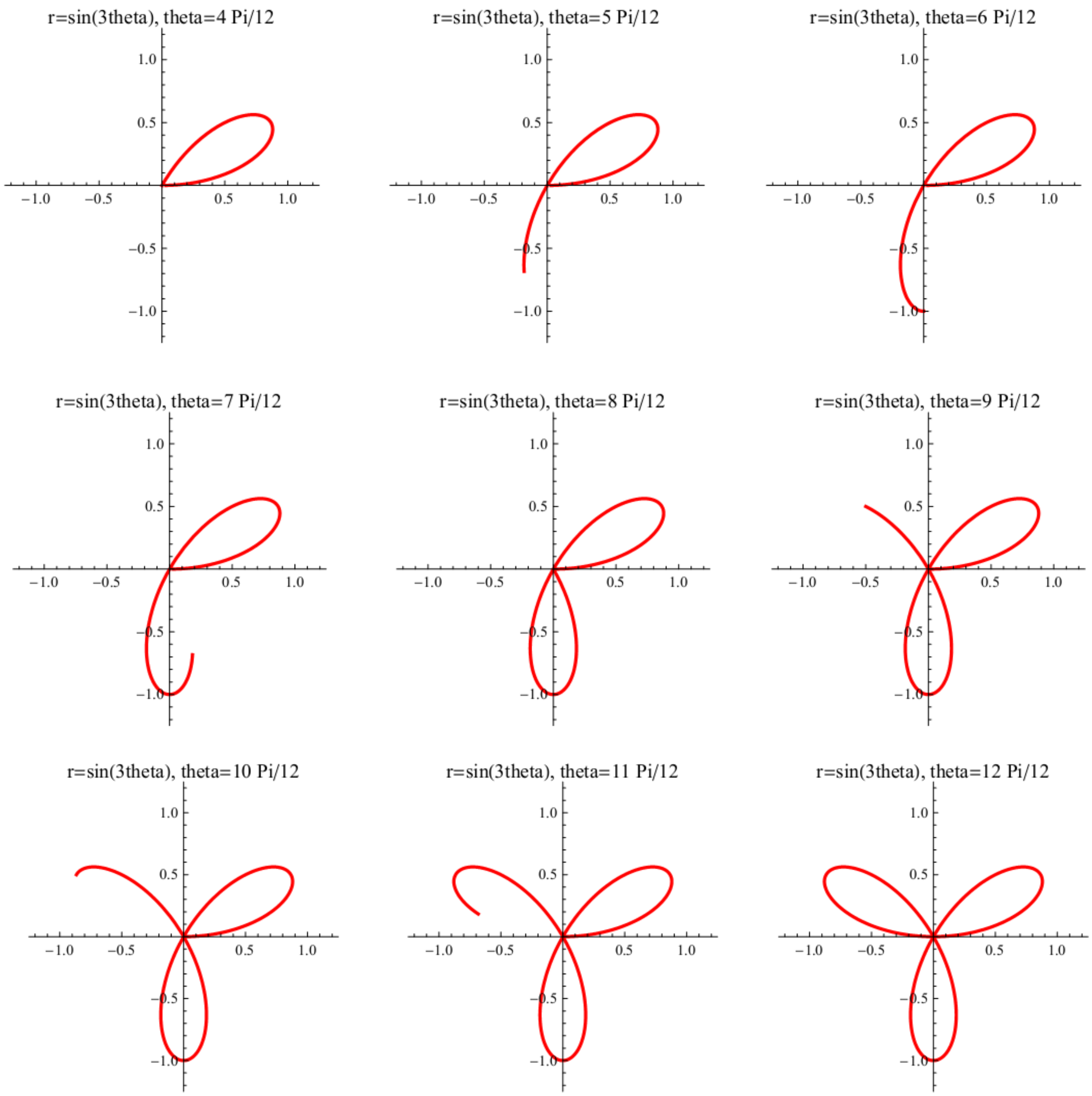


EXAMPLE: Sketch the curve r= sin(3ϴ); 0 ≤ ϴ ≤ π (three-leaved rose).

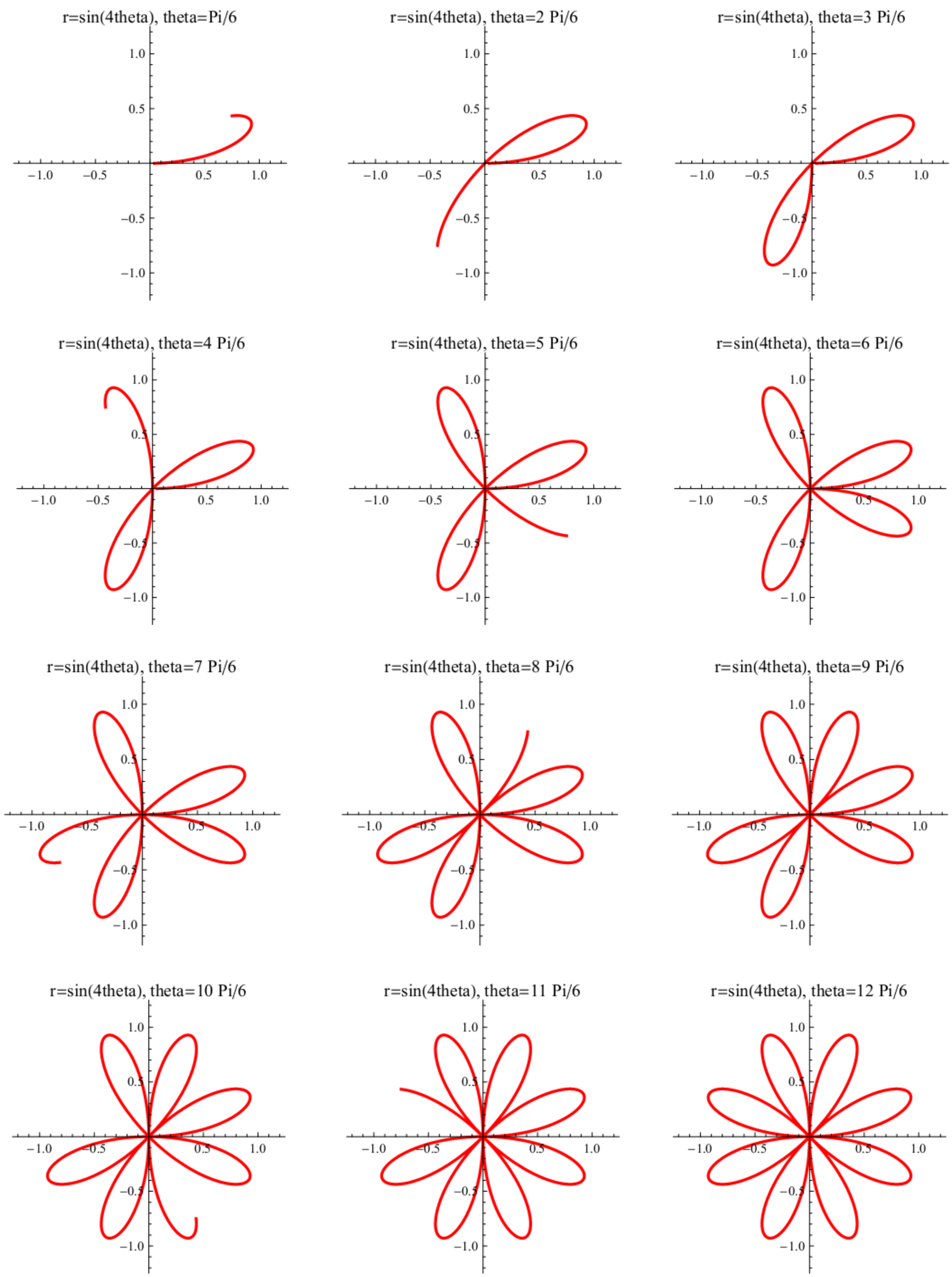
Solution: We have





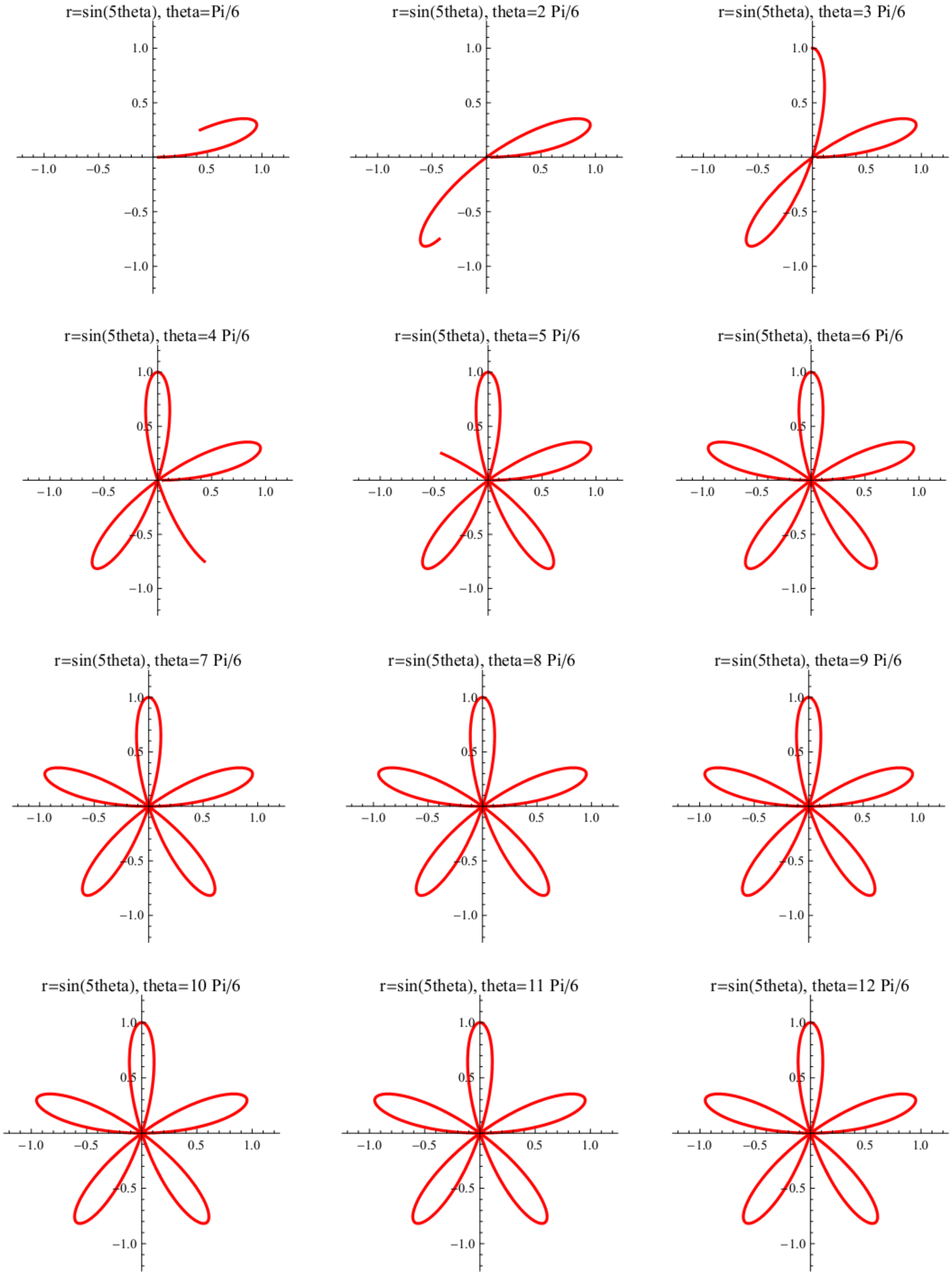
EXAMPLE: Sketch the curve r= sin(4Θ); 0 ≤ Θ ≤ 2π(eight-leaved rose).

Solution: We have



EXAMPLE: Sketch the curve r= sin(5Θ); 0 ≤ Θ ≤ 2π (five-leaved rose).

Solution: We have



Areas and Lengths in Polar Coordinates

In t his section we develop t he formula f o r t he area of a region whose boundary is given by a polar equation.

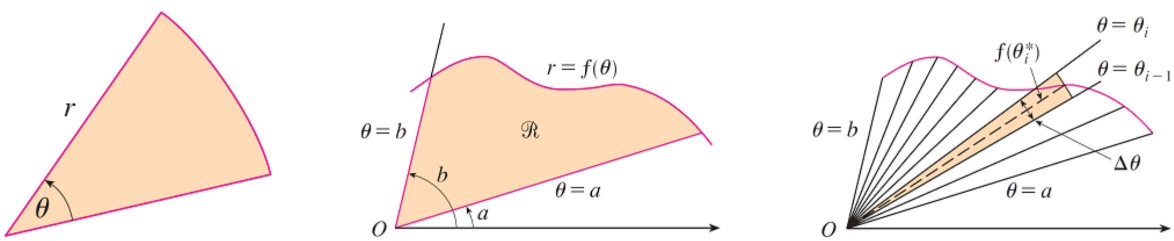
We need t o use t he formula f o r t he area o f a sect o r o f a circle

$$\boxed{A = \frac{1}{2}r^2\theta}$$

Where r is t he radius and θ is t he radian measure o f the central angle. Formula 1 follows from the fact that the area o f a sector is proportional to its central angle:

$$A = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2}r^2\theta$$

Let R be the region bounded by the polar curve r=f(θ) and by the rays θ=a and θ=b, where f is a positive continuous function and where $0 < b - a \leq 2\pi$.



We divide t he interval [a, b] in to subintervals with end points $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$ and equal width $\Delta\theta$. The rays $\theta=\theta_i$ then divide R in to n smaller regions with central angle $\Delta\theta=\theta_i - \theta_{i-1}$. If we choose θ^*_i in the

ith subinterval $[\theta_{i-1}, \theta_i]$, then the area ΔA_i of the ith region is approximated by the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta^*_i)$. Thus from Formula 1 we have

$$\Delta A_i \approx \frac{1}{2}[f(\theta_i^*)]^2\Delta\theta$$

and so an approximation to t he total area A o f R is

$$A \approx \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2\Delta\theta.$$

One can see that the approximation in (2) improves as n→ ∞. But t he sums in (2) are Riemann sums for the function

$$g(\theta) = \frac{1}{2}[f(\theta)]^2$$

so

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2 \Delta\theta = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta$$

It therefore appears plausible (and can in fact be proved) that the formula for the area A of the polar

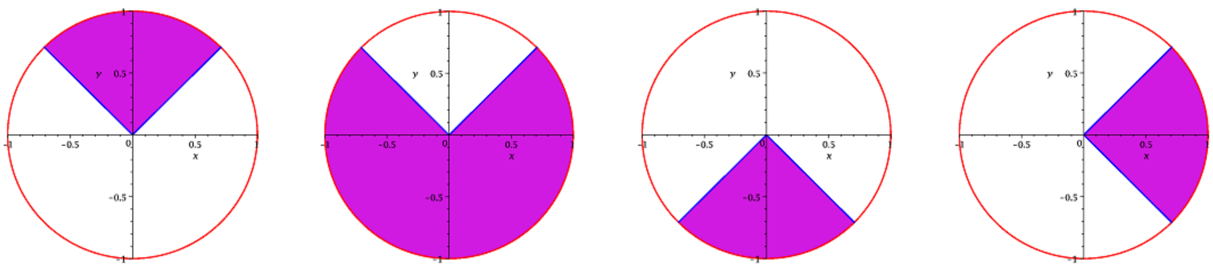
Region R is

$$\boxed{A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta}$$

This formula is often written as:

$$\boxed{A = \int_a^b \frac{1}{2} r^2 d\theta}$$

EXAMPL E: Find the area of each of the following regions:



(a) (b) (c) (d)

Solution:

(a) We have

$$A = \int_{\pi/4}^{3\pi/4} \frac{1}{2} \cdot 1^2 d\theta = \frac{1}{2}\int_{\pi/4}^{3\pi/4} d\theta = \frac{1}{2}\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = \frac{1}{2}\left(\frac{2\pi}{4}\right) = \frac{\pi}{4}$$

(b) We have

$$A=\int_{3\pi/4}^{2\pi+\pi/4}\frac{1}{2}\cdot 1^2 d\theta=\frac{1}{2}\int_{3\pi/4}^{2\pi+\pi/4}d\theta=\frac{1}{2}\left(2\pi+\frac{\pi}{4}-\frac{3\pi}{4}\right)=\frac{1}{2}\left(2\pi-\frac{2\pi}{4}\right)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$$

(c) We have

$$A=\int_{5\pi/4}^{7\pi/4}\frac{1}{2}\cdot 1^2 d\theta=\frac{1}{2}\int_{5\pi/4}^{7\pi/4}d\theta=\frac{1}{2}\left(\frac{7\pi}{4}-\frac{5\pi}{4}\right)=\frac{1}{2}\left(\frac{2\pi}{4}\right)=\frac{\pi}{4}$$

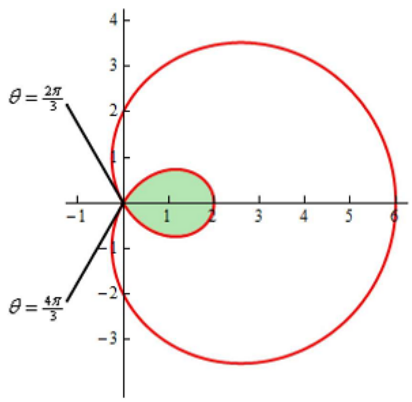
(d) We have

$$A=\int_{7\pi/4}^{2\pi+\pi/4}\frac{1}{2}\cdot 1^2 d\theta=\frac{1}{2}\int_{7\pi/4}^{2\pi+\pi/4}d\theta=\frac{1}{2}\left(2\pi+\frac{\pi}{4}-\frac{7\pi}{4}\right)=\frac{1}{2}\left(2\pi-\frac{6\pi}{4}\right)=\pi-\frac{3\pi}{4}=\frac{\pi}{4}$$

or

$$A=\int_{-\pi/4}^{\pi/4}\frac{1}{2}\cdot 1^2 d\theta=\frac{1}{2}\int_{-\pi/4}^{\pi/4}d\theta=\frac{1}{2}\left(\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right)=\frac{1}{2}\left(\frac{\pi}{4}+\frac{\pi}{4}\right)=\frac{\pi}{4}$$

EXAMPL E: Find the area of the inner loop of r= 2 + 4 cosθ .



Solution: We first find a and b:

$$2+4\cos\theta=0 \quad\Longrightarrow\quad \cos\theta=-\frac{1}{2} \quad\Longrightarrow\quad \theta=\frac{2\pi}{3},\ \frac{4\pi}{3}$$

There fore the area is

$$\begin{aligned} A &= \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(2+4\cos\theta)^2 d\theta = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(4+16\cos\theta+16\cos^2\theta)d\theta \\ &= \int_{2\pi/3}^{4\pi/3} (2+8\cos\theta+4(1+\cos(2\theta))d\theta = \int_{2\pi/3}^{4\pi/3} (6+8\cos\theta+4\cos(2\theta))d\theta \\ &= \Big[6\theta+8\sin\theta+2\sin(2\theta)\Big]_{2\pi/3}^{4\pi/3} = 4\pi-6\sqrt{3} \approx 2.174 \end{aligned}$$