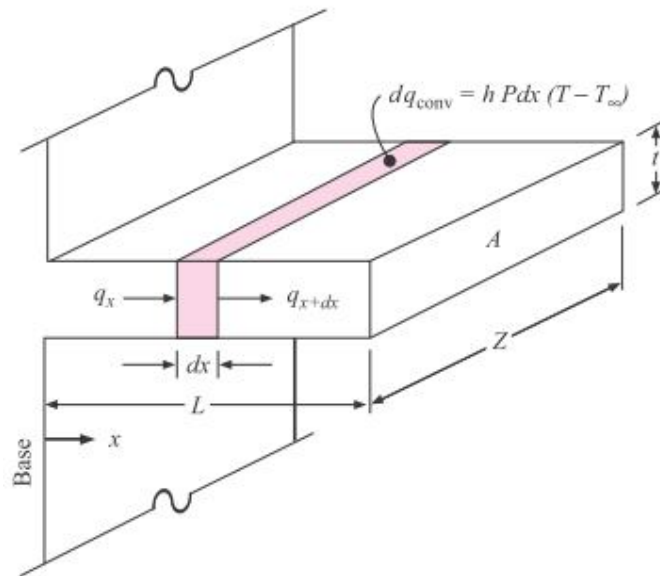


Conduction-Convection Systems

In heat-exchanger applications a finned-tube arrangement might be used to remove heat from a hot liquid. The heat transfer from the liquid to the finned tube is by convection. The extended-surface problem is a simple application of a conduction-convection systems. Consider the one-dimensional fin exposed to a surrounding fluid at a temperature T_∞ as shown in Figure below. The temperature of the base of the fin is T_0 .

Let P = perimeter = $2(Z + t)$



The energy balance on an element of the fin of thickness dx is

Energy in left face = energy out right face + energy lost by convection

$$\text{Energy in left face} = q_x = -kA \frac{dT}{dx}$$

$$\begin{aligned} \text{Energy out right face} &= q_{x+dx} = -kA \left. \frac{dT}{dx} \right|_{x+dx} \\ &= -kA \left(\frac{dT}{dx} + \frac{d^2T}{dx^2} dx \right) \end{aligned}$$

$$\text{Energy lost by convection} = h P dx (T - T_\infty)$$

$$\therefore -kA \frac{dT}{dx} = -kA \left[\frac{dT}{dx} + \frac{d^2T}{dx^2} dx \right] + h P dx (T - T_\infty), \quad \div dx \Rightarrow$$

$$\frac{d^2T}{dx^2} - \frac{hP}{kA} (T - T_{\infty}) = 0$$

$$\text{let } \theta = T - T_{\infty}, \quad \frac{dT}{dx} = \frac{d\theta}{dx} \Rightarrow$$

$$\frac{d^2T}{dx^2} = \frac{d^2\theta}{dx^2}$$

Let $m^2 = hP/kA$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

(D-Operator)

$$D^2 - 1 = 0 \Rightarrow D = \pm 1$$

\therefore The roots of the equation above are:

$$m_1 = +1, \quad m_2 = -1$$

The general solution is

$$\theta = C_1 e^{-mx} + C_2 e^{mx}$$

Several cases may be considered:

CASE 1: The fin is very long.

$$\text{B.C.1} \quad \text{at } x = 0 \quad T = T_0 \quad \theta = \theta_0$$

$$\text{B.C.2} \quad \text{at } x = \infty \quad T = T_{\infty} \quad \theta = 0$$

By applying the boundary conditions on the general equation above, we get

$$\frac{\theta}{\theta_0} = \frac{T - T_{\infty}}{T_0 - T_{\infty}} = e^{-mx}$$

CASE 2: The end of the fin is insulated so that $dT/dx = 0$ at $x = L$.

$$\text{B.C.1} \quad \text{at } x = 0 \quad T = T_0 \quad \theta = \theta_0$$

$$\text{B.C.2} \quad \text{at } x = L \quad dT/dx = 0 \quad d\theta/dx = 0$$

$$\text{From B.C.1} \quad \theta_0 = C_1 + C_2 \Rightarrow C_1 = \theta_0 - C_2$$

$$\text{From B.C.2} \quad \frac{dT}{dx} = -mC_1 e^{-mL} + mC_2 e^{mL} = 0$$

Subs. B.C.1 in B.C.2, we get

$$C_2 = \frac{\theta_0 e^{-mL}}{e^{mL} + e^{-mL}} \Rightarrow C_1 = \theta_0 - \frac{\theta_0 e^{-mL}}{e^{mL} + e^{-mL}}$$

Subs. C_1 & C_2 in the general equation, we obtain

$$\begin{aligned} \frac{\theta}{\theta_0} &= \frac{e^{-mx}}{1 + e^{-2mL}} + \frac{e^{mx}}{1 + e^{2mL}} \\ &= \frac{\cosh [m(L - x)]}{\cosh mL} \end{aligned}$$

CASE 3: The fin is of finite length and loses heat by convection from its end.

$$\text{B.C.1} \quad \text{at } x = 0 \quad T = T_0 \quad \theta = \theta_0$$

$$\text{B.C.2} \quad \text{at } x = L \quad -k \frac{d\theta}{dx} = h\theta$$

$$\text{From B.C.1} \quad \theta_0 = C_1 + C_2 \Rightarrow C_1 = \theta_0 - C_2$$

$$\text{From B.C.2} \quad -k(-mC_1 e^{-mL} + mC_2 e^{mL}) = h(C_1 e^{-mL} + C_2 e^{mL})$$

By applying C_1 & C_2 in the general equation, we get,

$$\frac{\theta}{\theta_0} = \frac{T - T_\infty}{T_0 - T_\infty} = \frac{\cosh m(L - x) + (h/mk) \sinh m(L - x)}{\cosh mL + (h/mk) \sinh mL}$$

Calculation of Heat Lost by the Fin

The heat lost by the fin can be calculated either by

$$q = -kA \left. \frac{dT}{dx} \right]_{x=0}$$

or by

$$q = \int_0^L hP(T - T_\infty) dx = \int_0^L hP \theta dx$$

FOR CASE 1:

$$\frac{\theta}{\theta_0} = e^{-mx}, \quad \frac{d\theta}{dx} = -m\theta_0 e^{-mx}, \quad \left. \frac{d\theta}{dx} \right|_{x=0} = -m\theta_0$$

$$\therefore -kA(-m\theta_0) = kAm\theta_0, \quad \text{For } q = kA\sqrt{\frac{hP}{kA}}\theta_0 \Rightarrow$$

$$q = \sqrt{hPkA} \theta_0$$

FOR CASE 2:

$$\frac{\theta}{\theta_0} = \frac{\cosh[m(L-x)]}{\cosh mL}, \quad \frac{d\theta}{dx} = \theta_0 \left(\frac{-me^{-mx}}{1+e^{-2mx}} + \frac{me^{mx}}{1+e^{2mx}} \right)$$

$$\left. \frac{d\theta}{dx} \right|_{x=0} = \theta_0 m \left(\frac{1}{1+e^{2mx}} - \frac{1}{1+e^{-2mx}} \right) \Rightarrow \frac{d\theta}{dx} = \theta_0 m \left(-\frac{e^{mL} - e^{-mL}}{e^{mx} + e^{-mL}} \right)$$

$$\therefore q = -kA\theta_0 m \left(-\frac{e^{mL} - e^{-mL}}{e^{mx} + e^{-mL}} \right) \Rightarrow$$

$$q = \sqrt{hPkA} \theta_0 \tanh mL$$

FOR CASE 3:

$$\frac{\theta}{\theta_0} = \frac{T - T_\infty}{T_0 - T_\infty} = \frac{\cosh m(L-x) + (h/mk) \sinh m(L-x)}{\cosh mL + (h/mk) \sinh mL}$$

The heat flow for this case is

$$q = \sqrt{hPkA} \theta_0 \frac{\sinh mL + (h/mk) \cosh mL}{\cosh mL + (h/mk) \sinh mL}$$