

## Chapter 02.02

### Differentiation of Continuous Functions

*After reading this chapter, you should be able to:*

1. *derive formulas for approximating the first derivative of a function,*
2. *derive formulas for approximating derivatives from Taylor series,*
3. *derive finite difference approximations for higher order derivatives, and*
4. *use the developed formulas in examples to find derivatives of a function.*

The derivative of a function at  $x$  is defined as

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

To be able to find a derivative numerically, one could make  $\Delta x$  finite to give,

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Knowing the value of  $x$  at which you want to find the derivative of  $f(x)$ , we choose a value of  $\Delta x$  to find the value of  $f'(x)$ . To estimate the value of  $f'(x)$ , three such approximations are suggested as follows.

#### Forward Difference Approximation of the First Derivative

From differential calculus, we know

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For a finite  $\Delta x$ ,

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

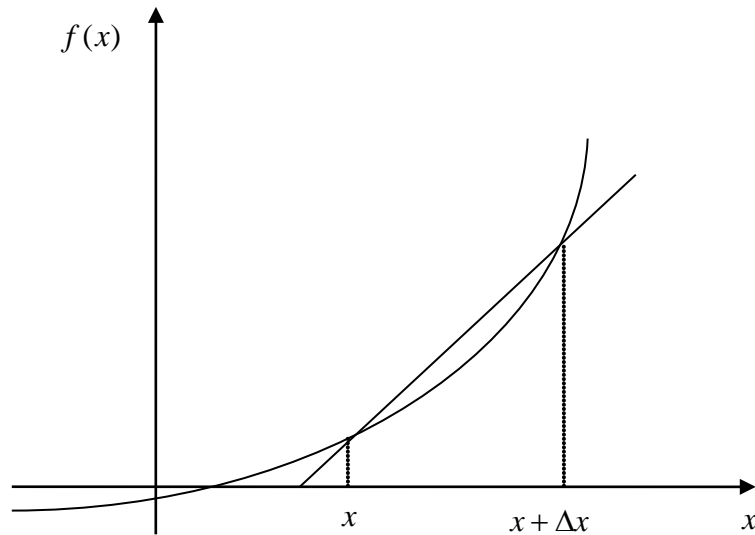
The above is the forward divided difference approximation of the first derivative. It is called forward because you are taking a point ahead of  $x$ . To find the value of  $f'(x)$  at  $x = x_i$ , we may choose another point  $\Delta x$  ahead as  $x = x_{i+1}$ . This gives

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{\Delta x}$$

$$= \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

where

$$\Delta x = x_{i+1} - x_i$$



**Figure 1** Graphical representation of forward difference approximation of first derivative.

### Example 1

The velocity of a rocket is given by

$$v(t) = 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100t} \right] - 9.8t, \quad 0 \leq t \leq 30$$

where  $v$  is given in m/s and  $t$  is given in seconds. At  $t = 16$  s,

- use the forward difference approximation of the first derivative of  $v(t)$  to calculate the acceleration. Use a step size of  $\Delta t = 2$  s.
- find the exact value of the acceleration of the rocket.
- calculate the absolute relative true error for part (b).

### Solution

$$(a) \quad a(t_i) \approx \frac{v(t_{i+1}) - v(t_i)}{\Delta t}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$t_{i+1} = t_i + \Delta t$$

$$= 16 + 2$$

$$= 18$$

$$\begin{aligned}
 a(16) &\approx \frac{v(18) - v(16)}{2} \\
 v(18) &= 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100(18)} \right] - 9.8(18) \\
 &= 453.02 \text{ m/s} \\
 v(16) &= 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100(16)} \right] - 9.8(16) \\
 &= 392.07 \text{ m/s}
 \end{aligned}$$

Hence

$$\begin{aligned}
 a(16) &\approx \frac{v(18) - v(16)}{2} \\
 &= \frac{453.02 - 392.07}{2} \\
 &= 30.474 \text{ m/s}^2
 \end{aligned}$$

(b) The exact value of  $a(16)$  can be calculated by differentiating

$$v(t) = 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100t} \right] - 9.8t$$

as

$$a(t) = \frac{d}{dt} [v(t)]$$

Knowing that

$$\begin{aligned}
 \frac{d}{dt} [\ln(t)] &= \frac{1}{t} \quad \text{and} \quad \frac{d}{dt} \left[ \frac{1}{t} \right] = -\frac{1}{t^2} \\
 a(t) &= 2000 \left( \frac{14 \times 10^4 - 2100t}{14 \times 10^4} \right) \frac{d}{dt} \left( \frac{14 \times 10^4}{14 \times 10^4 - 2100t} \right) - 9.8 \\
 &= 2000 \left( \frac{14 \times 10^4 - 2100t}{14 \times 10^4} \right) (-1) \left( \frac{14 \times 10^4}{(14 \times 10^4 - 2100t)^2} \right) (-2100) - 9.8 \\
 &= \frac{-4040 - 29.4t}{-200 + 3t} \\
 a(16) &= \frac{-4040 - 29.4(16)}{-200 + 3(16)} \\
 &= 29.674 \text{ m/s}^2
 \end{aligned}$$

(c) The absolute relative true error is

$$|\epsilon_t| = \left| \frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}} \right| \times 100$$

$$\begin{aligned}
 &= \left| \frac{29.674 - 30.474}{29.674} \right| \times 100 \\
 &= 2.6967\%
 \end{aligned}$$

### Backward Difference Approximation of the First Derivative

We know

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For a finite  $\Delta x$ ,

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If  $\Delta x$  is chosen as a negative number,

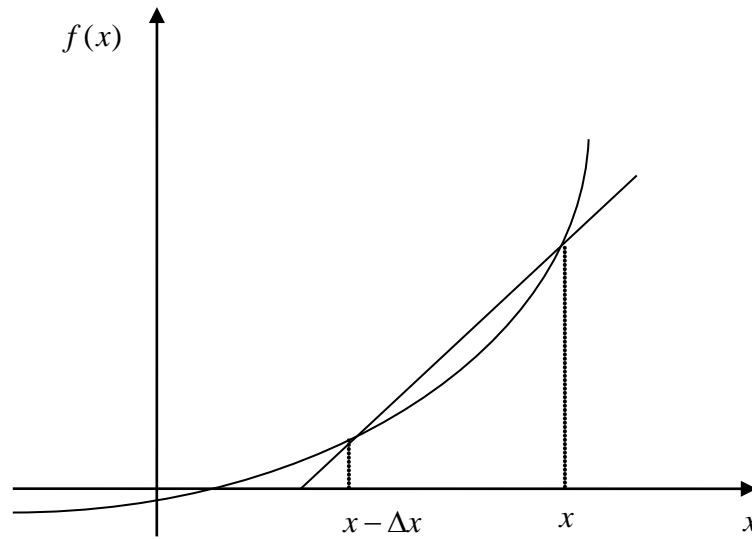
$$\begin{aligned}
 f'(x) &\approx \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \frac{f(x) - f(x - \Delta x)}{\Delta x}
 \end{aligned}$$

This is a backward difference approximation as you are taking a point backward from  $x$ . To find the value of  $f'(x)$  at  $x = x_i$ , we may choose another point  $\Delta x$  behind as  $x = x_{i-1}$ . This gives

$$\begin{aligned}
 f'(x_i) &\approx \frac{f(x_i) - f(x_{i-1})}{\Delta x} \\
 &= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}
 \end{aligned}$$

where

$$\Delta x = x_i - x_{i-1}$$



**Figure 2** Graphical representation of backward difference approximation of first derivative.

### Example 2

The velocity of a rocket is given by

$$v(t) = 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100t} \right] - 9.8t, 0 \leq t \leq 30$$

(a) Use the backward difference approximation of the first derivative of  $v(t)$  to calculate the acceleration at  $t = 16$  s. Use a step size of  $\Delta t = 2$  s.

(b) Find the absolute relative true error for part (a).

### Solution

$$a(t) \approx \frac{v(t_i) - v(t_{i-1})}{\Delta t}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$t_{i-1} = t_i - \Delta t$$

$$= 16 - 2$$

$$= 14$$

$$a(16) \approx \frac{v(16) - v(14)}{2}$$

$$\begin{aligned} v(16) &= 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100(16)} \right] - 9.8(16) \\ &= 392.07 \text{ m/s} \end{aligned}$$

$$v(14) = 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100(14)} \right] - 9.8(14)$$

$$= 334.24 \text{ m/s}$$

$$\begin{aligned} a(16) &\approx \frac{v(16) - v(14)}{2} \\ &= \frac{392.07 - 334.24}{2} \\ &= 28.915 \text{ m/s}^2 \end{aligned}$$

(b) The exact value of the acceleration at  $t = 16\text{s}$  from Example 1 is

$$a(16) = 29.674 \text{ m/s}^2$$

The absolute relative true error for the answer in part (a) is

$$\begin{aligned} |\epsilon_t| &= \left| \frac{29.674 - 28.915}{29.674} \right| \times 100 \\ &= 2.5584\% \end{aligned}$$

### Forward Difference Approximation from Taylor Series

Taylor's theorem says that if you know the value of a function  $f(x)$  at a point  $x_i$  and all its derivatives at that point, provided the derivatives are continuous between  $x_i$  and  $x_{i+1}$ , then

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \dots$$

Substituting for convenience  $\Delta x = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 + \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} - \frac{f''(x_i)}{2!}(\Delta x) + \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x)$$

The  $O(\Delta x)$  term shows that the error in the approximation is of the order of  $\Delta x$ .

Can you now derive from the Taylor series the formula for the backward divided difference approximation of the first derivative?

As you can see, both forward and backward divided difference approximations of the first derivative are accurate on the order of  $O(\Delta x)$ . Can we get better approximations? Yes, another method to approximate the first derivative is called the **central difference approximation of the first derivative**.

From the Taylor series

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f'''(x_i)}{3!}(\Delta x)^3 + \dots \quad (1)$$

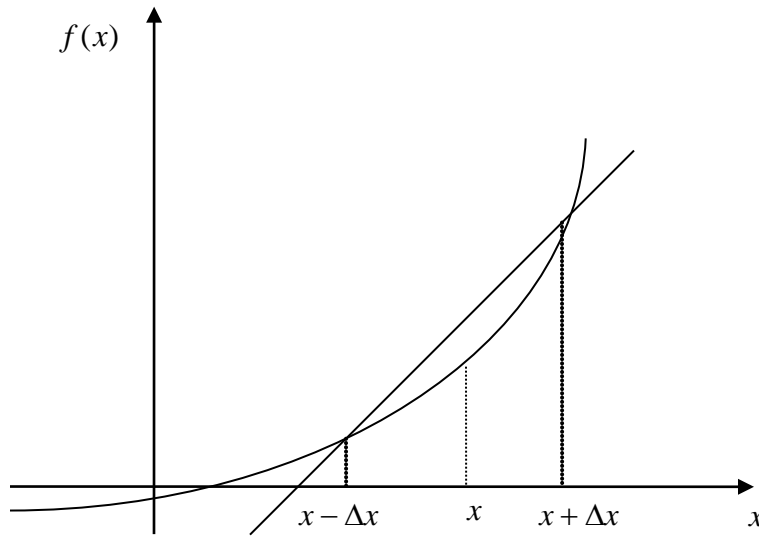
and

$$f(x_{i-1}) = f(x_i) - f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f'''(x_i)}{3!}(\Delta x)^3 + \dots \quad (2)$$

Subtracting Equation (2) from Equation (1)

$$\begin{aligned}
 f(x_{i+1}) - f(x_{i-1}) &= f'(x_i)(2\Delta x) + \frac{2f'''(x_i)}{3!}(\Delta x)^3 + \dots \\
 f'(x_i) &= \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} - \frac{f'''(x_i)}{3!}(\Delta x)^2 + \dots \\
 &= \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + O(\Delta x)^2
 \end{aligned}$$

hence showing that we have obtained a more accurate formula as the error is of the order of  $O(\Delta x)^2$ .



**Figure 3** Graphical representation of central difference approximation of first derivative.

### Example 3

The velocity of a rocket is given by

$$v(t) = 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100t} \right] - 9.8t, 0 \leq t \leq 30.$$

- Use the central difference approximation of the first derivative of  $v(t)$  to calculate the acceleration at  $t = 16$  s. Use a step size of  $\Delta t = 2$  s.
- Find the absolute relative true error for part (a).

### Solution

$$a(t_i) \approx \frac{v(t_{i+1}) - v(t_{i-1}))}{2\Delta t}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$t_{i+1} = t_i + \Delta t$$

$$= 16 + 2$$

$$= 18$$

$$t_{i-1} = t_i - \Delta t$$

$$= 16 - 2$$

$$= 14$$

$$a(16) \approx \frac{v(18) - v(14)}{2(2)}$$

$$= \frac{v(18) - v(14)}{4}$$

$$v(18) = 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100(18)} \right] - 9.8(18)$$

$$= 453.02 \text{ m/s}$$

$$v(14) = 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100(14)} \right] - 9.8(14)$$

$$= 334.24 \text{ m/s}$$

$$a(16) \approx \frac{v(18) - v(14)}{4}$$

$$= \frac{453.02 - 334.24}{4}$$

$$= 29.694 \text{ m/s}^2$$

(b) The exact value of the acceleration at  $t = 16 \text{ s}$  from Example 1 is

$$a(16) = 29.674 \text{ m/s}^2$$

The absolute relative true error for the answer in part (a) is

$$|\epsilon_t| = \left| \frac{29.674 - 29.694}{29.674} \right| \times 100$$

$$= 0.069157\%$$

The results from the three difference approximations are given in Table 1.

**Table 1** Summary of  $a(16)$  using different difference approximations

Type of difference approximation	$a(16)$ ( $\text{m/s}^2$ )	$ \epsilon_t \%$
Forward	30.475	2.6967
Backward	28.915	2.5584
Central	29.695	0.069157

Clearly, the central difference scheme is giving more accurate results because the order of accuracy is proportional to the square of the step size. In real life, one would not



know the exact value of the derivative – so how would one know how accurately they have found the value of the derivative? A simple way would be to start with a step size and keep on halving the step size until the absolute relative approximate error is within a pre-specified tolerance.

Take the example of finding  $v'(t)$  for

$$v(t) = 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100t} \right] - 9.8t$$

at  $t = 16$  using the backward difference scheme. Given in Table 2 are the values obtained using the backward difference approximation method and the corresponding absolute relative approximate errors.

**Table 2** First derivative approximations and relative errors for different  $\Delta t$  values of backward difference scheme.

$\Delta t$	$v'(t)$	$ \epsilon_a  \%$
2	28.915	
1	29.289	1.2792
0.5	29.480	0.64787
0.25	29.577	0.32604
0.125	29.625	0.16355

From the above table, one can see that the absolute relative approximate error decreases as the step size is reduced. At  $\Delta t = 0.125$ , the absolute relative approximate error is 0.16355%, meaning that at least 2 significant digits are correct in the answer.

### Finite Difference Approximation of Higher Derivatives

One can also use the Taylor series to approximate a higher order derivative. For example, to approximate  $f''(x)$ , the Taylor series is

$$f(x_{i+2}) = f(x_i) + f'(x_i)(2\Delta x) + \frac{f''(x_i)}{2!}(2\Delta x)^2 + \frac{f'''(x_i)}{3!}(2\Delta x)^3 + \dots \quad (3)$$

where

$$x_{i+2} = x_i + 2\Delta x$$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f'''(x_i)}{3!}(\Delta x)^3 \dots \quad (4)$$

where

$$x_{i-1} = x_i - \Delta x$$

Subtracting 2 times Equation (4) from Equation (3) gives

$$f(x_{i+2}) - 2f(x_{i+1}) = -f(x_i) + f''(x_i)(\Delta x)^2 + f'''(x_i)(\Delta x)^3 \dots$$

$$\begin{aligned}
 f''(x_i) &= \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{(\Delta x)^2} - f'''(x_i)(\Delta x) + \dots \\
 f''(x_i) &\approx \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{(\Delta x)^2} + O(\Delta x)
 \end{aligned} \tag{5}$$

**Example 4**

The velocity of a rocket is given by

$$v(t) = 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100t} \right] - 9.8t, 0 \leq t \leq 30$$

Use the forward difference approximation of the second derivative of  $v(t)$  to calculate the jerk at  $t = 16$  s. Use a step size of  $\Delta t = 2$  s.

**Solution**

$$j(t_i) \approx \frac{v(t_{i+2}) - 2v(t_{i+1}) + v(t_i)}{(\Delta t)^2}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$t_{i+1} = t_i + \Delta t$$

$$= 16 + 2$$

$$= 18$$

$$t_{i+2} = t_i + 2(\Delta t)$$

$$= 16 + 2(2)$$

$$= 20$$

$$j(16) \approx \frac{v(20) - 2v(18) + v(16)}{(2)^2}$$

$$\begin{aligned}
 v(20) &= 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100(20)} \right] - 9.8(20) \\
 &= 517.35 \text{ m/s}
 \end{aligned}$$

$$\begin{aligned}
 v(18) &= 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100(18)} \right] - 9.8(18) \\
 &= 453.02 \text{ m/s}
 \end{aligned}$$

$$\begin{aligned}
 v(16) &= 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100(16)} \right] - 9.8(16) \\
 &= 392.07 \text{ m/s}
 \end{aligned}$$

$$\begin{aligned}
 j(16) &\approx \frac{517.35 - 2(453.02) + 392.07}{4} \\
 &= 0.84515 \text{ m/s}^3
 \end{aligned}$$

The exact value of  $j(16)$  can be calculated by differentiating

$$v(t) = 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100t} \right] - 9.8t$$

twice as

$$a(t) = \frac{d}{dt} [v(t)] \text{ and}$$

$$j(t) = \frac{d}{dt} [a(t)]$$

Knowing that

$$\frac{d}{dt} [\ln(t)] = \frac{1}{t} \text{ and}$$

$$\frac{d}{dt} \left[ \frac{1}{t} \right] = -\frac{1}{t^2}$$

$$\begin{aligned} a(t) &= 2000 \left( \frac{14 \times 10^4 - 2100t}{14 \times 10^4} \right) \frac{d}{dt} \left( \frac{14 \times 10^4}{14 \times 10^4 - 2100t} \right) - 9.8 \\ &= 2000 \left( \frac{14 \times 10^4 - 2100t}{14 \times 10^4} \right) (-1) \left( \frac{14 \times 10^4}{(14 \times 10^4 - 2100t)^2} \right) (-2100) - 9.8 \\ &= \frac{-4040 - 29.4t}{-200 + 3t} \end{aligned}$$

Similarly it can be shown that

$$\begin{aligned} j(t) &= \frac{d}{dt} [a(t)] \\ &= \frac{18000}{(-200 + 3t)^2} \\ j(16) &= \frac{18000}{[-200 + 3(16)]^2} \\ &= 0.77909 \text{ m/s}^3 \end{aligned}$$

The absolute relative true error is

$$\begin{aligned} |\epsilon_i| &= \left| \frac{0.77909 - 0.84515}{0.77909} \right| \times 100 \\ &= 8.4797\% \end{aligned}$$

The formula given by Equation (5) is a forward difference approximation of the second derivative and has an error of the order of  $O(\Delta x)$ . Can we get a formula that has a better accuracy? Yes, we can derive the central difference approximation of the second derivative. The Taylor series is

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f'''(x_i)}{3!}(\Delta x)^3 + \frac{f^{(4)}(x_i)}{4!}(\Delta x)^4 + \dots \quad (6)$$

where

$$x_{i+1} = x_i + \Delta x$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f'''(x_i)}{3!}(\Delta x)^3 + \frac{f^{(4)}(x_i)}{4!}(\Delta x)^4 - \dots \quad (7)$$

where

$$x_{i-1} = x_i - \Delta x$$

Adding Equations (6) and (7), gives

$$\begin{aligned} f(x_{i+1}) + f(x_{i-1}) &= 2f(x_i) + f''(x_i)(\Delta x)^2 + f^{(4)}(x_i)\frac{(\Delta x)^4}{12} + \dots \\ f''(x_i) &= \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1}))}{(\Delta x)^2} - \frac{f^{(4)}(x_i)(\Delta x)^2}{12} + \dots \\ &= \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1}))}{(\Delta x)^2} + O(\Delta x)^2 \end{aligned}$$

### Example 5

The velocity of a rocket is given by

$$v(t) = 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100t} \right] - 9.8t, \quad 0 \leq t \leq 30,$$

(a) Use the central difference approximation of the second derivative of  $v(t)$  to calculate the jerk at  $t = 16$  s. Use a step size of  $\Delta t = 2$  s.

### Solution

The second derivative of velocity with respect to time is called jerk. The second order approximation of jerk then is

$$j(t_i) \approx \frac{v(t_{i+1}) - 2v(t_i) + v(t_{i-1}))}{(\Delta t)^2}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$t_{i+1} = t_i + \Delta t$$

$$= 16 + 2$$

$$= 18$$

$$t_{i-1} = t_i - \Delta t$$

$$= 16 - 2$$

$$= 14$$

$$j(16) \approx \frac{v(18) - 2v(16) + v(14)}{(2)^2}$$

$$\begin{aligned}v(18) &= 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100(18)} \right] - 9.8(18) \\ &= 453.02 \text{ m/s}\end{aligned}$$

$$\begin{aligned}v(16) &= 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100(16)} \right] - 9.8(16) \\ &= 392.07 \text{ m/s}\end{aligned}$$

$$\begin{aligned}v(14) &= 2000 \ln \left[ \frac{14 \times 10^4}{14 \times 10^4 - 2100(14)} \right] - 9.8(14) \\ &= 334.24 \text{ m/s}\end{aligned}$$

$$\begin{aligned}j(16) &\approx \frac{v(18) - 2v(16) + v(14)}{(2)^2} \\ &= \frac{453.02 - 2(392.07) + 334.24}{4} \\ &= 0.77969 \text{ m/s}^3\end{aligned}$$

The absolute relative true error is

$$\begin{aligned}|\epsilon_t| &= \left| \frac{0.77908 - 0.77969}{0.77908} \right| \times 100 \\ &= 0.077992\%\end{aligned}$$