Chapter 11.02
Continuous Fourier Series

For a function with period T, a continuous Fourier series can be expressed as [1-5]

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kw_0 t) + b_k \sin(kw_0 t) \tag{1}$$

The unknown Fourier coefficients a_0, a_k and b_k can be computed as

$$a_0 = \left(\frac{1}{T}\right)\int_0^T f(t)dt \tag{2}$$

Thus, a_0 can be interpreted as the "average" function value between the period interval $[0, T]$.

$$a_k = \left(\frac{2}{T}\right)\int_0^T f(t)\cos(kw_0 t)dt \tag{3}$$

$$\equiv a_{-k} \quad \text{(hence } a_k \text{ is an "even" function)}$$

$$b_k = \left(\frac{2}{T}\right)\int_0^T f(t)\sin(kw_0 t)dt \tag{4}$$

$$\equiv -b_{-k} \quad \text{(hence } b_k \text{ is an "odd" function)}$$

Derivation of formulas for a_0, a_k and b_k

Integrating both sides of Equation 1 with respect to time, one gets

$$\int_0^T f(t)dt = \int_0^T a_0 dt + \int_0^T \sum_{k=1}^{\infty} a_k \cos(kw_0 t)dt + \int_0^T \sum_{k=1}^{\infty} b_k \sin(kw_0 t)dt \tag{5}$$

The second and third terms on the right hand side of the above equations are both zeros, due to the result stated in Equation (1) of Chapter 11.01.
Thus,

$$\int_0^T f(t)dt = [a_0 t]_0^T \tag{6}$$

$$= a_0 T$$

Hence,

$$a_0 = \left(\frac{1}{T}\right)\int_0^T f(t)dt \tag{7}$$

Now, if both sides of Equation (1) are multiplied by $\sin(mw_0t)$ and then integrated with respect to time, one obtains

$$\int_0^T f(t)\times\sin(mw_0t)dt = \int_0^T a_0\sin(mw_0t)dt + \int_0^T\sum_{k=1}^{\infty}a_k\cos(kw_0t)\sin(mw_0t)dt + \int_0^T\sum_{k=1}^{\infty}b_k\sin(kw_0t)\sin(mw_0t)dt \tag{8}$$

Due to Equations (1) and (3) of Chapter 11.01, the first and second terms on the right hand side (RHS) of Equation (8) are zero.

Due to Equation (4) of Chapter 11.01, the third RHS term of Equation (8) is also zero, with the exception when $k = m$, which will become (by referring to Equation (2) of Chapter 11.01)

$$\int_0^T f(t)\sin(kw_0t)dt = 0 + 0 + \int_0^T b_k\sin^2(kw_0t)dt = b_k\times\frac{T}{2} \tag{9}$$

Thus,

$$b_k = \left(\frac{2}{T}\right)\int_0^T f(t)\sin(kw_0t)dt$$

Similar derivation can be used to obtain a_k, as shown in Equation (3)

A FORTRAN Program for finding Fourier Coefficients a_0, a_k, and b_k

Based upon the derived formulas for a_0, a_k and b_k (shown in Equations 2-4), a FORTRAN/MATLAB computer program has been developed. (The program is available at http://numericalmethods.eng.usf.edu/simulations/mtl/11fft/f_coeff_final.m)

Example 1

Using the continuous Fourier series to approximate the following periodic function ($T = 2\pi$ seconds) shown in Figure 1.

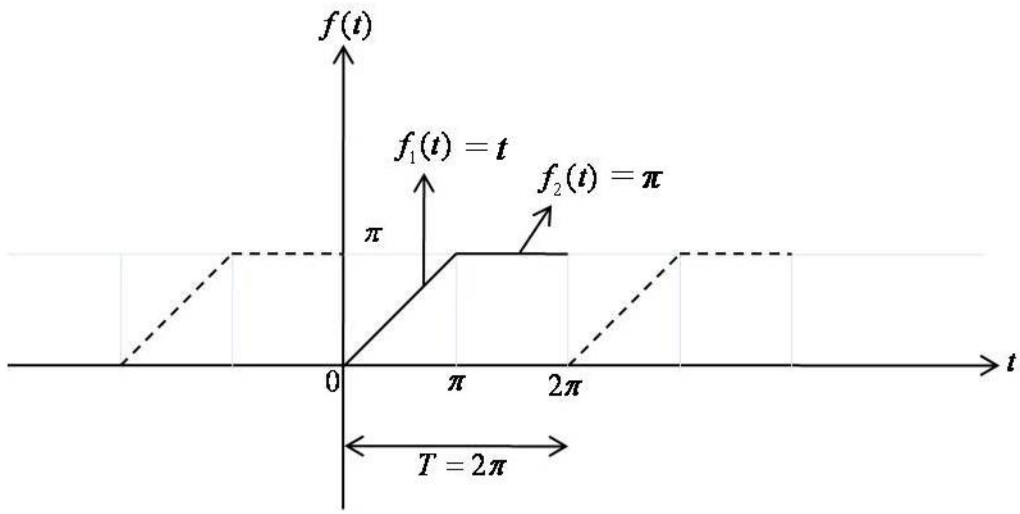

Figure 1 A Periodic Function (Between 0 and 2π).

$$f(t) = \begin{cases} t \text{ for } 0 < t \le \pi \\ \pi \text{ for } \pi \le t < 2\pi \end{cases}$$

Specifically, find the Fourier coefficients $a_0, a_1, ..., a_8$ and $b_1, ..., b_8$.

Solution

The unknown Fourier coefficients a_0, a_k and b_k can be computed based on Equations (2–4); as following:

$$a_0 = \left(\frac{1}{T}\right)\int_0^{2\pi} f(t)dt$$

$$a_0 = \frac{1}{(2\pi)} \times \left\{\int_0^{\pi} t dt + \int_{\pi}^{2\pi} \pi dt\right\}$$

$$a_0 = 2.35619$$

$$a_k = \left(\frac{2}{T}\right)\int_0^{T=2\pi} f(t)\cos(kw_0 t)dt$$

$$a_k = \left(\frac{2}{2\pi}\right) \times \left\{\int_0^{\pi} t\cos\left(k \times \frac{2\pi}{T} \times t\right)dt + \int_{\pi}^{2\pi} \pi \times \cos\left(k \times \frac{2\pi}{T} \times t\right)dt\right\}$$

$$a_k = \left(\frac{1}{\pi}\right) \times \left\{\int_0^{\pi} t\cos(kt)dt + \int_{\pi}^{2\pi} \pi\cos(kt)dt\right\}$$

The “integration by part” formula can be utilized to compute the first integral on the right-hand-side of the above equation.

For $k = 1,2,....,8,$ the Fourier coefficients a_k can be computed as

$$a_1 = -0.6366257003116296$$

$$a_2 = -5.070352857678721 \times 10^{-6} \approx 0$$

$$a_3 = -0.07074100153210318$$

$$a_4 = -5.070320092569666 \times 10^{-6} \approx 0$$
$$a_5 = -0.025470225589332522$$
$$a_6 = -5.070265333302604 \times 10^{-6} \approx 0$$
$$a_7 = -0.0012997664818977102$$
$$a_8 = -5.070188612604695 \times 10^{-6} \approx 0$$

Similarly,

$$b_k = \left(\frac{2}{T}\right)\int_0^{2\pi} f(t)\sin(kw_0t)dt$$

$$b_k = \left(\frac{1}{\pi}\right)\times\left\{\int_0^{\pi} t\sin(kt)dt + \int_{\pi}^{2\pi} \pi\sin(kt)dt\right\}$$

For $k = 1,2,....,8,$ the Fourier coefficients b_k can be computed as

$$b_1 = -0.9999986528958207$$
$$b_2 = -0.4999993232285269$$
$$b_3 = -0.3333314439509194$$
$$b_4 = -0.24999804122384547$$
$$b_5 = -0.19999713794872364$$
$$b_6 = -0.1666635603759553$$
$$b_7 = -0.14285324664625462$$
$$b_8 = -0.12499577981019251$$

Any periodic function $f(t)$, such as the one shown in Figure 1 can be represented by the Fourier series as

$$f(t) = a_0 + \sum_{k=1}^{\infty}\{a_k\cos(kw_0t) + b_k\sin(kw_0t)\}$$

where a_0, a_k and b_k have already been computed (for $k = 1,2,....,8,$);

and $w_0 = 2\pi f$

$$= \frac{2\pi}{T}$$

$$= \frac{2\pi}{2\pi}$$

$$= 1$$

Thus, for $k = 1$, one obtains

$$\bar{f}_1(t) \approx a_0 + a_1\cos(t) + b_1\sin(t)$$

For $k = 1 \to 2$, one obtains

$$\bar{f}_2(t) \approx a_0 + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t)$$

For $k = 1 \to 4$, one obtains

$$\bar{f}_4(t) \approx a_0 + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t) + a_3\cos(3t) + b_3\sin(3t) + a_4\cos(4t) + b_4\sin(4t)$$

Plots for $\bar{f}_1(t)$, $\bar{f}_2(t)$ and $\bar{f}_4(t)$ are shown in Figure 2.

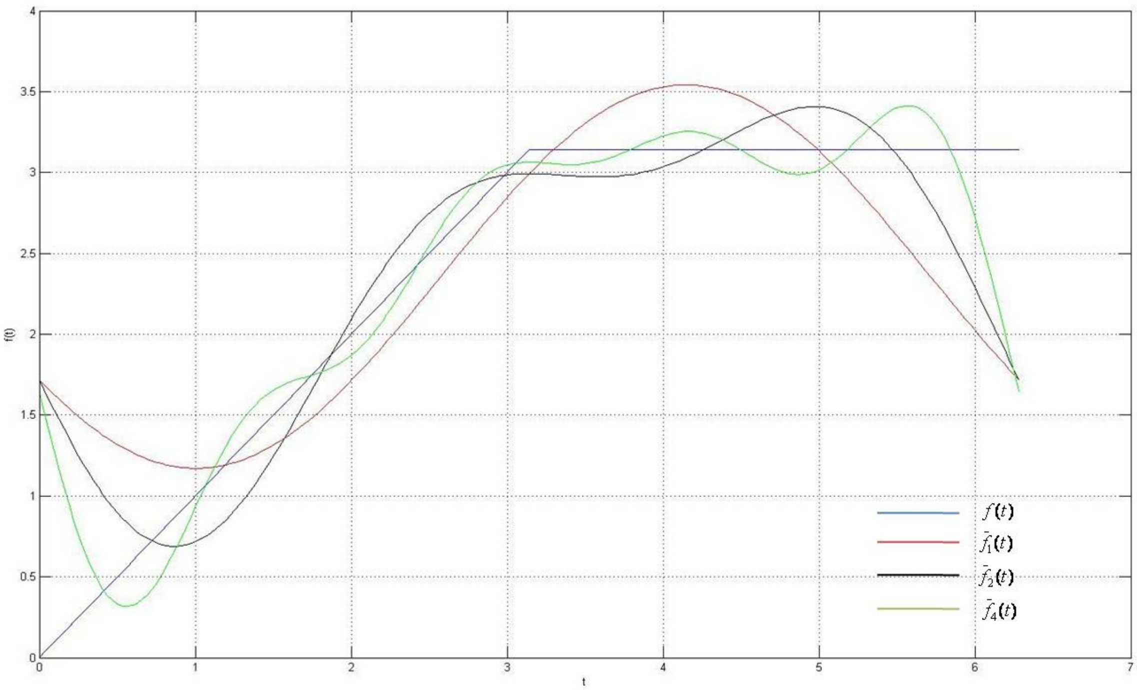

Figure 2 Fourier Approximated Functions (for Example 1).

It can be observed from Figure 2 that as more terms are included in the Fourier series, the approximated Fourier functions are more closely resemble the original periodic function as shown in Figure 1.

Example 2

The periodic triangular wave function $f(t)$ is defined as

$$f(t) = \begin{cases} \dfrac{-\pi}{2} \text{ for } -\pi < t < \dfrac{-\pi}{2} \\ -t \text{ for } \dfrac{-\pi}{2} < t < \dfrac{\pi}{2} \\ \dfrac{-\pi}{2} \text{ for } \dfrac{\pi}{2} < t < \pi \end{cases}$$

Find the Fourier coefficients $a_0, a_1, ..., a_8$ and $b_1, ..., b_8$ and approximate the periodic triangular wave function by the Fourier series.

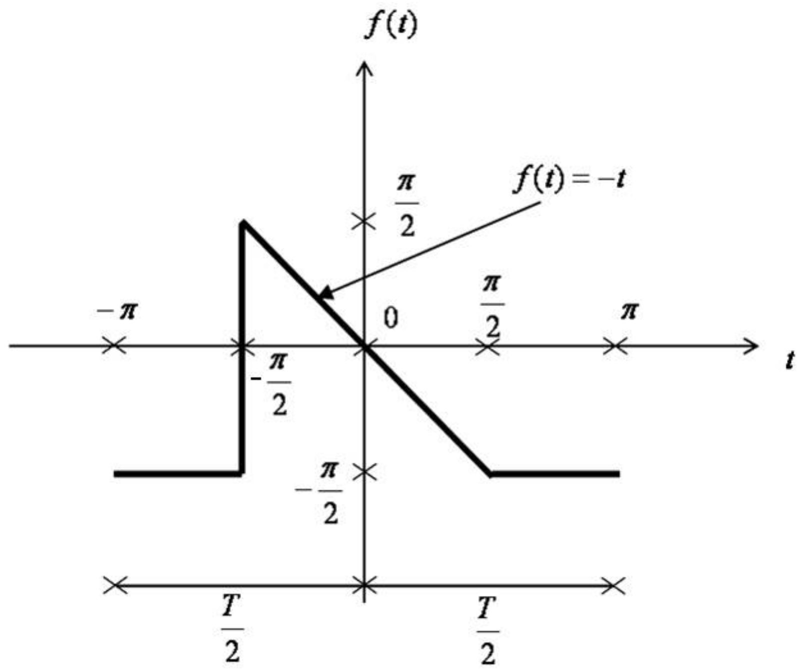

Figure 3 Periodic triangular wave function for Example 2.

Solution

The unknown Fourier Coefficients a_0, a_k and b_k can be computed based on Equations (2-4) as follows

$$a_0 = \left(\frac{1}{T}\right)\int_{-\pi}^{\pi} f(t)dt$$

$$a_0 = \frac{1}{(2\pi)} \times \left\{ \int_{-\pi}^{-\frac{\pi}{2}} \left(-\frac{\pi}{2}\right)dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-t)dt + \int_{\frac{\pi}{2}}^{\pi} \left(-\frac{\pi}{2}\right)dt \right\}$$

$$a_0 = -0.78539753$$

$$a_k = \left(\frac{2}{T}\right)\int_{-\pi}^{\pi} f(t)\cos(kw_0 t)dt$$

where

$$w_0 = \frac{2\pi}{T}$$

$$= \frac{2\pi}{2\pi}$$

$$= 1$$

Hence,

$$a_k = \left(\frac{2}{T}\right)\int_{-\pi}^{\pi} f(t)\cos(kt)dt$$

or

$$a_k = \left(\frac{2}{2\pi}\right)\left\{\int_{-\pi}^{-\frac{\pi}{2}}\left(-\frac{\pi}{2}\right)\cos(kt)dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(-t)\cos(kt)dt + \int_{\frac{\pi}{2}}^{\pi}\left(-\frac{\pi}{2}\right)\cos(kt)dt\right\}$$

Similarly,

$$b_k = \left(\frac{2}{T}\right)\int_{-\pi}^{\pi} f(t)\sin(kw_0t)dt = \left(\frac{2}{T}\right)\int_{-\pi}^{\pi} f(t)\sin(kt)dt$$

or,

$$b_k = \left(\frac{2}{2\pi}\right)\left\{\int_{-\pi}^{-\frac{\pi}{2}}\left(-\frac{\pi}{2}\right)\sin(kt)dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(-t)\sin(kt)dt + \int_{\frac{\pi}{2}}^{\pi}\left(-\frac{\pi}{2}\right)\sin(kt)dt\right\}$$

The "integration by part" formula can be utilized to compute the second integral on the right-hand-side of the above equations for a_k and b_k.

For $k = 1,2,...8,$ the Fourier coefficients a_k and b_k can be computed and summarized as following in Table 1

Table 1 Fourier coefficients a_k and b_k for various k values.

k	a_k	b_k
1	0.999997	-0.63661936
2	0.00	-0.49999932
3	-0.3333355	0.07073466
4	0.00	0.2499980
5	0.1999968	-0.02546389
6	0.00	-0.16666356
7	-0.14285873	0.0126991327
8	0.00	0.12499578

The periodic function (shown in Example 1) can be approximated by Fourier series as

$$f(t) = a_0 + \sum_{k=1}^{\infty}\{a_k \cos(kt) + b_k \sin(kt)\}$$

Thus, for $k = 1$, one obtains:

$$\bar{f}_1(t) = a_0 + a_1\cos(t) + b_1\sin(t)$$

For $k = 1 \to 2,$ one obtains:

$$\bar{f}_2(t) = a_0 + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t)$$

Similarly, for $k = 1 \to 4,$ one has:

$$\bar{f}_4(t) = a_0 + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t) + a_3\cos(3t) + b_3\sin(3t) + a_4\cos(4t) + b_4\sin(4t)$$

Plots for functions $\bar{f}_1(t),\ \bar{f}_2(t)$ and $\bar{f}_4(t)$ are shown in Figure 4.

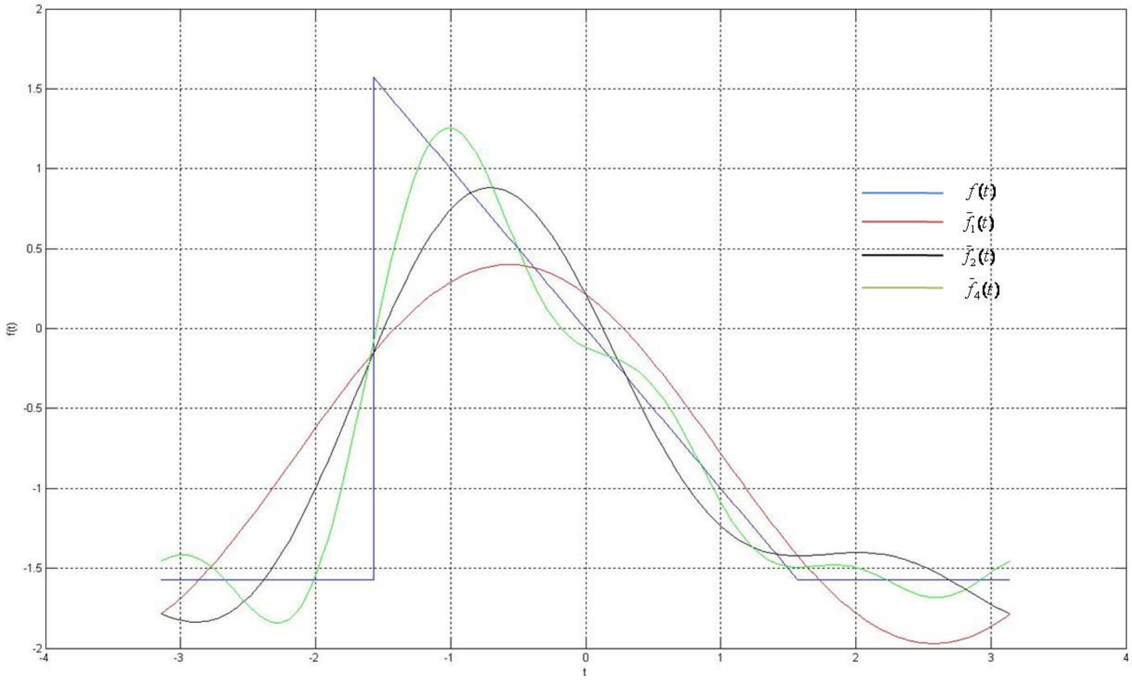

Figure 4 Fourier approximated functions for Example 2.

It can be observed from Figure 4 that as more terms are included in the Fourier series, the approximated Fourier functions closely resemble the original periodic function.

Complex Form of the Fourier Series

Using Euler's identity, $e^{ix} = \cos(x) + i\sin(x),$ and $e^{-ix} = \cos(x) - i\sin(x),$ the sine and cosine can be expressed in the exponential form as

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} = \text{"}odd\text{"}\ function,\ \text{since}\ \sin(x) = -\sin(-x) \tag{10}$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} = \text{"}even\text{"}\ function,\ \text{since}\ \cos(x) = \cos(-x) \tag{11}$$

Thus, the Fourier series (expressed in Equation 1) can be converted into the following form

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \left(\frac{e^{ikw_0t} + e^{-ikw_0t}}{2} \right) + b_k \left(\frac{e^{ikw_0t} - e^{-ikw_0t}}{2i} \right) \tag{12}$$

or

$$f(t) = a_0 + \sum_{k=1}^{\infty} e^{ikw_0t} \left(\frac{a_k}{2} + \frac{b_k}{2i} * \frac{i}{i} \right) + e^{-ikw_0t} \left(\frac{a_k}{2} - \frac{b_k}{2i} * \frac{i}{i} \right)$$

or, since $i^2 = -1,$ one obtains

$$f(t) = a_0 + \sum_{k=1}^{\infty} e^{ikw_0t} \left(\frac{a_k - ib_k}{2} \right) + e^{-ikw_0t} \left(\frac{a_k + ib_k}{2} \right) \tag{13}$$

Define the following constants

$$\tilde{C}_0 \equiv a_0 \tag{14}$$

$$\tilde{C}_k \equiv \frac{a_k - ib_k}{2} \tag{15}$$

Hence:

$$\tilde{C}_{-k} \equiv \frac{a_{-k} - ib_{-k}}{2} \tag{16}$$

Using the even and odd properties shown in Equations (3) and (4) respectively, Equation (16) becomes

$$\tilde{C}_{-k} \equiv \frac{a_k + ib_k}{2} \tag{17}$$

Substituting Equations (14), (15), (17) into Equation (13), one gets

$$\begin{aligned} f(t) &= \tilde{C}_0 + \sum_{k=1}^{\infty} \tilde{C}_k e^{ikw_0 t} + \sum_{k=1}^{\infty} \tilde{C}_{-k} e^{-ikw_0 t} \\ &= \sum_{k=0}^{\infty} \tilde{C}_k e^{ikw_0 t} + \sum_{k=-1}^{-\infty} \tilde{C}_k e^{ikw_0 t} \\ &= \sum_{k=0}^{\infty} \tilde{C}_k e^{ikw_0 t} + \sum_{k=-\infty}^{-1} \tilde{C}_k e^{ikw_0 t} \\ &= \sum_{k=-\infty}^{\infty} \tilde{C}_k e^{ikw_0 t} \end{aligned} \tag{18}$$

The coefficient $\tilde{C}_k$ can be computed, by substituting Equations (3) and (4) into Equation (15) to obtain

$$\tilde{C}_k = \left(\frac{1}{2}\right)\left(\frac{2}{T}\right)\left\{\int_0^T f(t)\cos(kw_0 t)dt - i\int_0^T f(t)\sin(kw_0 t)dt\right\} \tag{19}$$

$$= \left(\frac{1}{T}\right)\left\{\int_0^T f(t) \times \left[\cos(kw_0 t) - i\sin(kw_0 t)\right]dt\right\}$$

Substituting Equations (10, 11) into the above equation, one gets

$$\tilde{C}_k = \left(\frac{1}{T}\right)\left\{\int_0^T f(t) \times \left[\frac{e^{ikw_0 t} + e^{-ikw_0 t}}{2} - i \times \frac{e^{ikw_0 t} - e^{-ikw_0 t}}{2i}\right]dt\right\}$$

$$= \left(\frac{1}{T}\right)\left\{\int_0^T f(t) \times e^{-ikw_0 t} dt\right\} \tag{20}$$

Thus, Equations (18) and (20) are the equivalent complex version of Equations (1)-(4).

References

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