08.06
Shooting Method
for Ordinary Differential Equations

After reading this chapter, you should be able to

1. *learn the shooting method algorithm to solve boundary value problems, and*
2. *apply shooting method to solve boundary value problems.*

What is the shooting method?

Ordinary differential equations are given either with initial conditions or with boundary conditions. Look at the problem below.

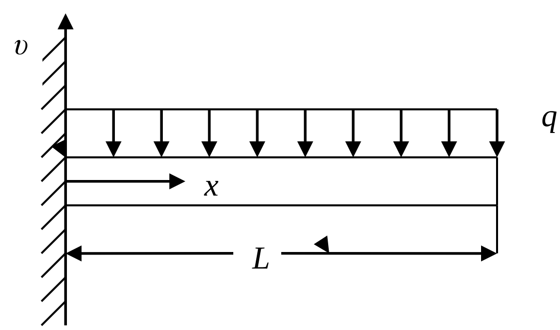

Figure 1 A cantilevered uniformly loaded beam.

To find the deflection υ as a function of location x, due to a uniform load q, the ordinary differential equation that needs to be solved is

$$\frac{d^2\upsilon}{dx^2} = \frac{q}{2EI}(L-x)^2 \tag{1}$$

where

L is the length of the beam,
E is the Young's modulus of the beam, and
I is the second moment of area of the cross-section of the beam.

Two conditions are needed to solve the problem, and those are

$$\upsilon(0) = 0$$

$$\frac{d\upsilon}{dx}(0) = 0 \qquad \text{(2a,b)}$$

as it is a cantilevered beam at $x = 0$. These conditions are *initial conditions* as they are given at an initial point, $x = 0$, so that we can find the deflection along the length of the beam.

Now consider a similar beam problem, where the beam is simply supported on the two ends

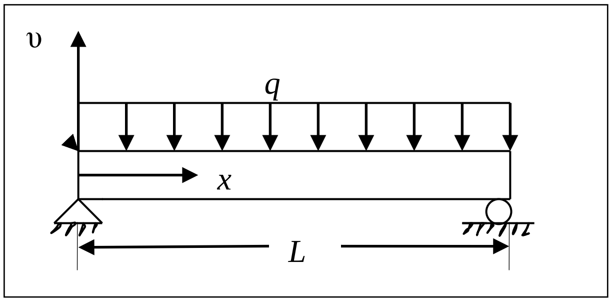

Figure 2 A simply supported uniformly loaded beam.

To find the deflection υ as a function of x due to the uniform load q, the ordinary differential equation that needs to be solved is

$$\frac{d^2\upsilon}{dx^2} = \frac{qx}{2EI}(x - L) \qquad (3)$$

Two conditions are needed to solve the problem, and those are

$$\upsilon(0) = 0$$
$$\upsilon(L) = 0 \qquad \text{(4a,b)}$$

as it is a simply supported beam at $x = 0$ and $x = L$. These conditions are *boundary conditions* as they are given at the two boundaries, $x = 0$ and $x = L$.

The shooting method

The shooting method uses the same methods that were used in solving initial value problems. This is done by assuming initial values that would have been given if the ordinary differential equation were an initial value problem. The boundary value obtained is then compared with the actual boundary value. Using trial and error or some scientific approach, one tries to get as close to the boundary value as possible. This method is best explained by an example.

Take the case of a pressure vessel that is being tested in the laboratory to check its ability to withstand pressure. For a thick pressure vessel of inner radius a and outer radius b, the differential equation for the radial displacement u of a point along the thickness is given by

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \qquad (5)$$

Assume that the inner radius $a = 5"$ and the outer radius $b = 8"$, and the material of the pressure vessel is ASTM36 steel. The yield strength of this type of steel is 36 ksi. Two strain gages that are bonded tangentially at the inner and the outer radius measure the normal tangential strain in the pressure vessel as

$$\in_{t/r=a} = \quad 0.00077462$$

$$\in_{t/r=b} = \quad 0.00038462 \tag{6ab}$$

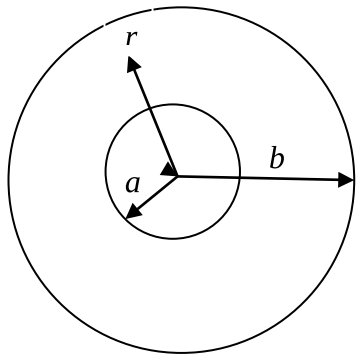

Figure 3 Cross-sectional geometry of a pressure vessel.

at the maximum needed pressure. Since the radial displacement and tangential strain are related simply by

$$\in_t = \frac{u}{r}, \tag{7}$$

then

$$u\big|_{r=a} = \quad 0.00077462 \times 5 \quad = \quad 0.0038731''$$
$$u\big|_{r=b} = \quad 0.00038462 \times 8 \quad = \quad 0.0030770'' \tag{8}$$

Starting with the ordinary differential equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0,\ u(5) = 0.0038731, u(8) = 0.0030770$$

Let

$$\frac{du}{dr} = w \tag{9}$$

Then

$$\frac{dw}{dr} + \frac{1}{r}w - \frac{u}{r^2} = 0 \tag{10}$$

giving us two first order differential equations as

$$\frac{du}{dr} = w,\ u(5) = 0.0038731''$$
$$\frac{dw}{dr} = -\frac{w}{r} + \frac{u}{r^2},\ w(5) = not\ known \tag{11a,b}$$

Let us assume

$$w(5) = \frac{du}{dr}(5) \approx \frac{u(8) - u(5)}{8 - 5} = -0.00026538$$

Set up the initial value problem.

$$\frac{du}{dr} = w = f_1(r, u, w), u(5) = 0.0038731''$$

$$\frac{dw}{dr} = -\frac{w}{r} + \frac{u}{r^2} = f_2(r,u,w), w(5) = -0.00026538 \tag{12a,b}$$

Using Euler's method,

$$u_{i+1} = u_i + f_1(r_i, u_i, w_i)h$$
$$w_{i+1} = w_i + f_2(r_i, u_i, w_i)h \tag{13a,b}$$

Let us consider 4 segments between the two boundaries, $r = 5''$ and $r = 8''$, then

$$h = \frac{8-5}{4} = 0.75''$$

$i = 0,\ r_0 = 5,\ u_0 = 0.0038731'', w_0 = -0.00026538$

$$\begin{aligned}u_1 &= u_0 + f_1(r_0, u_0, w_0)h\\ &= 0.0038371 + f_1(5, 0.0038371, -0.00026538)(0.75)\\ &= 0.0038371 + (-0.00026538)(0.75)\\ &= 0.0036741''\end{aligned}$$

$$\begin{aligned}w_1 &= w_0 + f_2(r_0, u_0, w_0)h\\ &= -0.00026538 + f_2(5, 0.0038731, -0.00026538)(0.75)\\ &= -0.00026538 + \left(-\frac{-0.00026538}{5} + \frac{0.0038371}{5^2}\right)(0.75)\\ &= -0.00010938\end{aligned}$$

$i = 1,\ r_1 = r_0 + h = 5 + 0.75 = 5.75'',$

$u_1 = 0.0036741'',\ w_1 = -0.00010940$

$$\begin{aligned}u_2 &= u_1 + f_1(r_1, u_1, w_1)h\\ &= 0.0036741 + f_1(5.75, 0.0036741, -0.00010938)(0.75)\\ &= 0.0036741 + (-0.00010938)(0.75)\\ &= 0.0035920''\end{aligned}$$

$$\begin{aligned}w_2 &= w_1 + f_2(r_1, u_1, w_1)h\\ &= -0.00010938 + f_2(5.75, 0.0036741, -0.00010938)(0.75)\\ &= -0.00010938 + (0.00013015)(0.75)\\ &= -0.000011769\end{aligned}$$

$i = 2, r_2 = r_1 + h = 5.75 + 0.75 = 6.5''$

$u_2 = 0.0035920'', w_2 = -0.000011785$

$$\begin{aligned}u_3 &= u_2 + f_1(r_2, u_2, w_2)h\\ &= 0.0035920 + f_1(6.5, 0.0035920, -0.000011769)(0.75)\\ &= 0.0035920 + (-0.000011769)(0.75)\\ &= 0.0035832''\end{aligned}$$

$$\begin{aligned}w_3 &= w_2 + f_2(r_2, u_2, w_2)h\\ &= -0.000011769 + f_2(6.5, 0.0035920 - 0.000011769)(0.75)\end{aligned}$$

$$= -0.000011769 + (0.000086829)(0.75)$$
$$= 0.000053352$$

$$i = 3,\ r_3 = r_2 + h = 6.50 + 0.75 = 7.25''$$
$$u_3 = 0.0035832'',\ w_3 = 0.000053352$$
$$u_4 = u_3 + f_1(r_3, u_3, w_3)h$$
$$= 0.0035832 + f_1(7.25, 0.0035832, 0.000053352)(0.75)$$
$$= 0.0035832 + (0.000053352)(0.75)$$
$$= 0.0036232''$$
$$w_4 = w_3 + f_2(r_3, u_3, w_3)h$$
$$= -0.000011785 + f_2(7.25, 0.0035832, -0.000053352)(0.75)$$
$$= 0.000053352 + (0.000060811)(0.75)$$
$$= 0.000098961$$

At
$$r = r_4 = r_3 + h = 7.25 + 0.75 = 8''$$
we have
$$u_4 = u(8) \approx 0.0036232''$$

While the given value of this boundary condition is
$$u_4 = u(8) = 0.003070''$$

Let us assume a new value for $\frac{du}{dr}(5)$. Based on the first assumed value, maybe using twice the value of initial guess.

$$w(5) = 2\frac{du}{dr}(5) \approx 2\frac{u(8) - u(5)}{8 - 5} = 2(-0.00026538) = -0.00053076$$

Using $h = 0.75$, and Euler's method, we get
$$u_4 = u(8) \approx 0.0029665''$$
While the given value of this boundary condition is
$$u_4 = u(8) = 0.0030770''$$

Can we use the results obtained from the two previous iterations to get a better estimate of the assumed initial condition of $\frac{du}{dr}(5)$? One method is to use linear interpolation on the obtained data for the two assumed values of $\frac{du}{dr}(5)$.

With
$$\frac{du}{dr}(5) \approx -0.00026538,$$
we obtained

$$u(8) \approx 0.0036232'', \text{ and}$$

with

$$\frac{du}{dr}(5) \approx -0.00053076,$$

we obtained

$$u(8) \approx 0.0029665''$$

so a better starting value of $\frac{du}{dr}(5)$ knowing that the actual value at

$$u(8) = 0.00030770'',$$

we get

$$\frac{du}{dr}(5) \approx \frac{-0.00053076 - (-0.00026538)}{0.0029645 - 0.0036232}(0.0030770 - 0.0036232) + (-0.00026538)$$
$$= -0.00048611$$

Using $h = 0.75''$, and repeating the Euler's method with

$$w(5) = -0.00048611,$$

we get

$$u_4 = u(8) \approx 0.0030769''$$

while the actual given value of this boundary condition is

$$u(8) = 0.0030770''.$$

In this case, this value coincides with the actual value of $u(8)$. If that were not the case, one would continue to use linear interpolation to refine the value of u_4 till one gets close to the actual value of $u(8)$. Note that the step size and the numerical method used would influence the accuracy for the obtained values. For the last case, the values are as follows

$$u_0 = u(5) = 0.0038731''$$
$$u_1 = u(5.75) \approx 0.0035085''$$
$$u_2 = u(6.50) \approx 0.0032858''$$
$$u_3 = u(7.25) \approx 0.0031518''$$
$$u_4 = u(8.00) \approx 0.0030770''$$

See Figure 4 for the comparison of the results with different initial guesses of the slope.

Using $h = 0.75''$ and Runge-Kutta 4th order method,

$$u_1 = u(5) = 0.0038731''$$
$$u_2 = u(5.75) \approx 0.0035554''$$
$$u_3 = u(6.50) \approx 0.0033341''$$
$$u_4 = u(7.25) \approx 0.00317923''$$
$$u_5 = u(8) \approx 0.0030723''$$

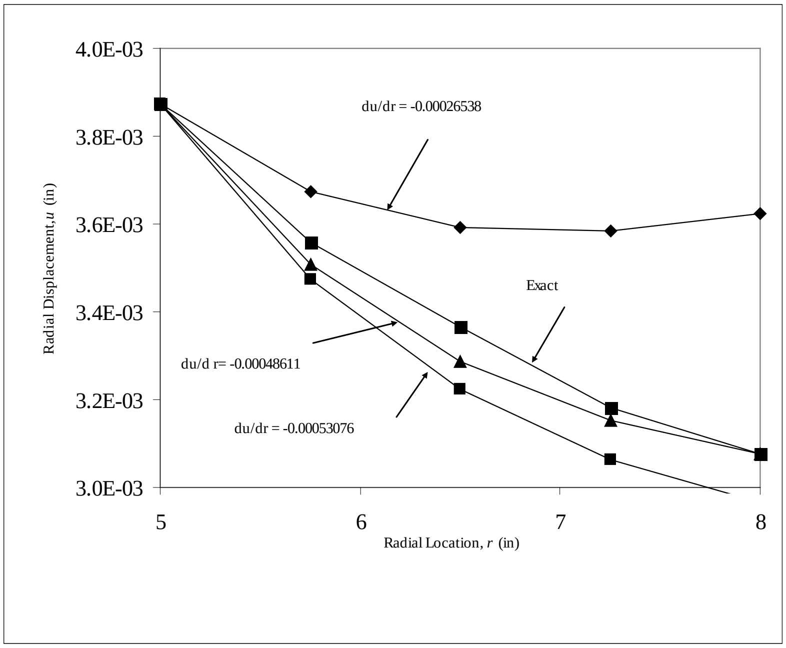

Figure 4 Comparison of results with different initial guesses of slope

Table 1 shows the comparison of the results obtained using Euler's, Runge-Kutta and exact methods.

Table 1 Comparison of Euler and Runge-Kutta results with exact results.

r (in)	Exact (in)	Euler (in)	$\lvert\varepsilon_t\rvert$(%)	Runge-Kutta (in)	$\lvert\varepsilon_t\rvert$(%)
5	3.8731×10^{-3}	3.8731×10^{-3}	0.0000	3.8731×10^{-3}	0.0000
5.75	3.5567×10^{-3}	3.5085×10^{-3}	1.3731	3.5554×10^{-3}	3.5824×10^{-2}
6.5	3.3366×10^{-3}	3.2858×10^{-3}	1.5482	3.3341×10^{-3}	7.4037×10^{-2}
7.25	3.1829×10^{-3}	3.1518×10^{-3}	9.8967×10^{-1}	3.1792×10^{-3}	1.1612×10^{-1}
8	3.0770×10^{-3}	3.0770×10^{-3}	1.9500×10^{-3}	3.0723×10^{-3}	1.5168×10^{-1}