Angular frequency

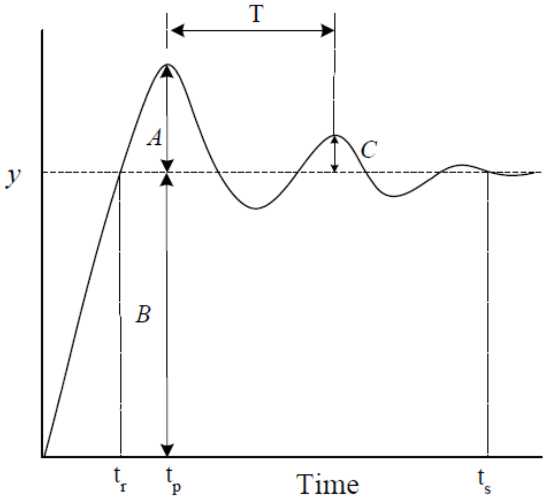

Fig.27 characteristics of second-order under-damped response

From eq. of the response y(t)

$$y(t) = a \sin \omega t$$

$$\omega = \frac{\sqrt{1-\zeta^2}}{\tau} \quad and\ \tau = \frac{1}{\omega_n} \quad , \omega = \omega_n\sqrt{1-\zeta^2}$$

Cycle period T_d

$$T_d = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Time of first peak, t_p

The maximum value of response occur at minimum time at t>0 and sine function =0 .this is called the peak time.

$$y(t) = kA\left[1 - \frac{e^{\frac{-\zeta}{\tau}t}}{\sqrt{1-\zeta^2}}\left(\sin(\frac{\sqrt{1-\zeta^2}}{\tau}t + \varphi)\right)\right]$$

$$y'(t) = \frac{-kA}{\sqrt{1-\zeta^2}}\left[e^{\frac{-\zeta}{\tau}t}\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}t+\varphi\right).\frac{\sqrt{1-\zeta^2}}{\tau}+\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t+\varphi\right)e^{\frac{-\zeta}{\tau}t}\left(\frac{-\zeta}{\tau}\right)\right]$$

$$y'(t) = \frac{-kA}{\tau\sqrt{1-\zeta^2}}e^{\frac{-\zeta}{\tau}t}\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)$$

$$y'(t) = 0 \rightarrow \sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right) = 0 \rightarrow \frac{\sqrt{1-\zeta^2}}{\tau}t_p = n\pi$$

The first peak at n=1

$$t_p = \frac{\pi\tau}{\sqrt{1-\zeta^2}}$$

Overshoot

It is a measure of how much the response exceeds the ultimate value following a step change and is expressed as the ratio a/B

At t= t_p:

$$y(t) = kA\left[1-\frac{e^{\frac{-\zeta}{\tau}t_p}}{\sqrt{1-\zeta^2}}\left(\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t_p+\varphi\right)\right)\right]$$

But $t_p = \frac{\pi\tau}{\sqrt{1-\zeta^2}}$

$$y(t) = kA\left(1-\frac{e^{\frac{-\zeta}{\tau}\frac{\pi\tau}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\left(\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}\frac{\pi\tau}{\sqrt{1-\zeta^2}}+\varphi\right)\right)\right)$$

$$y(t) = kA\left(1-\frac{e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}(\sin(\pi+\varphi))\right)$$

$$y(t) = kA\left(1-\frac{e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}(-sin\varphi)\right)$$

$$y(t) = kA\left(1 + \frac{e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sqrt{1-\zeta^2}\right)$$

$$y(t) = kA\left(1 + e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}\right) \quad max.peak$$

$$y(\infty) = Ak\ , overshoot = Ak\left(1 + e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}\right) - Ak$$

$$\therefore overshoot = Ake^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$overshoot\% = \frac{a}{B} = \frac{Ak\ e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}}{Ak} \times 100\% = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\%$$

Subsidence ratio (SR)

$$SR = \frac{a}{b} = \frac{c}{d}$$

$$a = Ake^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$b = Ake^{\frac{-\zeta}{\sqrt{1-\zeta^2}}(3\pi)}$$

$$SR = \frac{a}{b} = e^{\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Decay ratio(DR)

$$DR = \frac{1}{SR} = \frac{b}{a} = \frac{d}{c} = e^{\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Optimum ,DR=1/4,SR=4 ,ζ=0.215 0.2<ζ<0.5

Optimum response at ζ=0.215 ,DR=1/4

Rise time,(t_r)

This is a time required for the response to first reach at ultimate value .

$$y(t) = kA\left(1 - \frac{e^{\frac{-\zeta}{\tau}t}}{\sqrt{1-\zeta^2}}\left(\sin(\frac{\sqrt{1-\zeta^2}}{\tau}t + \varphi)\right)\right)$$

$$at\ t = t_r\ , y(t) = Ak$$

$$\therefore\ \frac{e^{\frac{-\zeta}{\tau}t_r}}{\sqrt{1-\zeta^2}}\left(\sin(\frac{\sqrt{1-\zeta^2}}{\tau}t_r + \varphi)\right) = 0$$

$$\left(\sin(\frac{\sqrt{1-\zeta^2}}{\tau}t_r + \varphi)\right) = 0$$

$$\frac{\sqrt{1-\zeta^2}}{\tau}t_r + \varphi = n\pi$$

n=1 at first peak

$$\frac{\sqrt{1-\zeta^2}}{\tau}t_r = \pi - \varphi$$

$$t_R = \frac{\tau(\pi - \varphi)}{\sqrt{1-\zeta^2}} = \frac{\tau}{\sqrt{1-\zeta^2}}\left(\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$