Newtons Laws of Motion

1- Every body (or part of a body) continues in a state or rest, or uniform velocity (i.e. linear, angular or both) unless acted upon by a (net or resultant) force.

$$\Sigma F_x = 0,\ \Sigma F_y = 0,\ \Sigma F_z = 0 \quad \text{AND}$$
$$\Sigma M_x = 0,\ \Sigma M_y = 0,\ \Sigma M_z = 0$$

2- The change of velocity (i.e the acceleration) is proportional to the NET force (or moment) acting on the body.

$F = ma$
$T = I\alpha$

m – mass
a – linear acceleration
I – Moment of Inertia
α – Angular acceleration

3- To every action (i.e force or moment) there is always an equal (and opposite) reaction, force or moment, (at the point of application whether OR not the body is accelerating).

Generalised Hookes Laws

The displacement at a point is directly proportional to the load applied at that or any other point in the member or structure.

i.e.

Stress = k x strain

For direct stress and strains

$\sigma = E\epsilon$

where:

E - Young modulas (Modulus of elasticity)

For shear stresses and strains

$\tau = G\gamma$

where

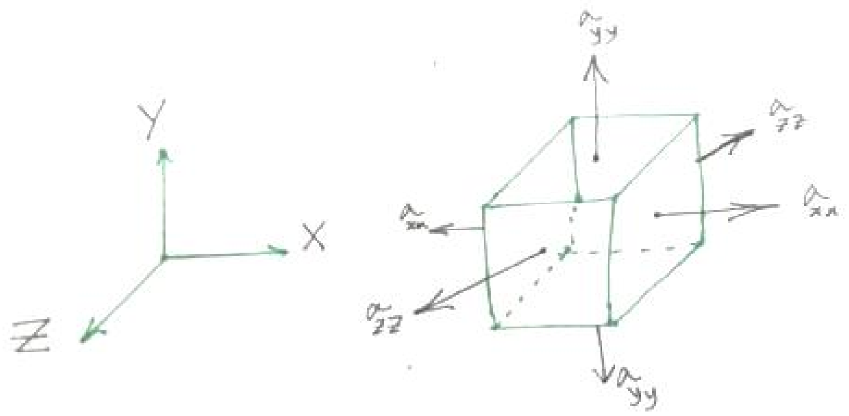
G - shear modulas (Modulus of rigidity)

σ_{xx} produces ϵ_{xx} in the x direction
AND $\nu \epsilon_{xx}$ in the Y direction
AND $\nu \epsilon_{xx}$ in the Z direction

i.e. the stress σ_{xx} produces $\frac{\sigma_{xx}}{E}$ in the x direction
AND $\frac{\nu}{E}\sigma_{xx}$ in the Y direction
AND $\frac{\nu}{E}\sigma_{xx}$ in the Z direction.

ALL THESE FROM ONE STRESS

For (3) three direct stresses σ_{xx}, σ_{yy}, σ_{zz} AND no shear stresses applied, on one element



The total strains in each direction are:

$$\left.\begin{aligned}
\epsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \\
\epsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \\
\epsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{yy} + \sigma_{xx})\right]
\end{aligned}\right\} \text{plane stress}$$

Rearranging the Equations gives

$$\left.\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_{xx} + \nu(\epsilon_{yy} + \epsilon_{zz})\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_{yy} + \nu(\epsilon_{zz} + \epsilon_{xx})\right] \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_{zz} + \nu(\epsilon_{xx} + \epsilon_{yy})\right]
\end{aligned}\right\} \text{plane strain}$$

This illustrates the Principle of SUPERPOSITION for linear elastic stress & strain. "If the material of a body is subjected to two or more stresses. The strain produced by one of the stresses is the same as if the other stresses are absent"

We can thus estimate the pattern of stress and strain from one set of external loads (forces or moments) and add the results to those determind from any other sets of load which may be acting at the same time, or even later.

To complete the discription of strain, the shear strains are:

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G}$$

from which the shear stresses [NO POISSON'S RATIO EFFECT] are:

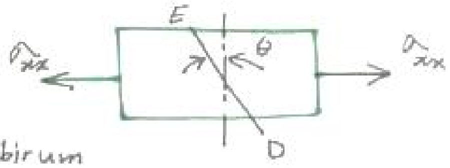
$$\tau_{xy} = G\gamma_{xy}, \quad \tau_{yz} = G\gamma_{yz}, \quad \tau_{zx} = G\gamma_{zx}$$

Typical value of (γ) for metal is 0.3

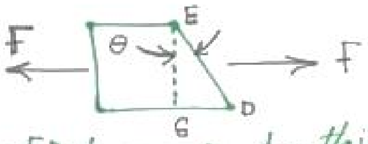
Stress Transformation Equations

Element subjected to a single direct stress:

Consider stresses on a plane ED inclined at θ.

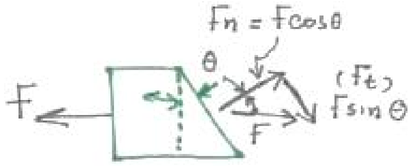


Considering the force equilibirum of part of element gives:



$$\cos\theta = \frac{EG}{ED}$$

ie area of the inclined plane, $A = ED \times t \longrightarrow t$ – thickness



$$\frac{EG}{ED} = \cos\theta$$

$$\therefore ED = \frac{EG}{\cos\theta}$$

Resolving (F) into two Components gives:

$$F_n = F\cos\theta$$
$$F_t = F\sin\theta$$

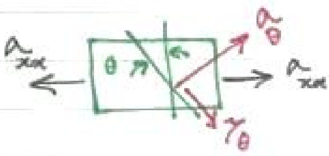
$$\therefore \sigma_\theta = \frac{F_n}{A} = \frac{F\cos\theta}{\left(\frac{EG}{\cos\theta}\right)\times t} = \frac{F\cos^2\theta}{EG\times t} = \frac{F}{EG\times t}\cdot\cos^2\theta$$

$$\sigma_\theta = \sigma_{xx}\cos^2\theta = \frac{\sigma_{xx}}{2}(1-\cos 2\theta)$$

AND

$$\tau_\theta = \frac{F_t}{A} = \frac{F\sin\theta}{A} = \frac{F\sin\theta}{\left(\frac{EG}{\cos\theta}\right)\times t} = \frac{F\sin\theta\cos\theta}{EG\times t} = \frac{F}{EG\times t}\sin\theta\cos\theta$$

$$\tau_\theta = \sigma_{xx}\sin\theta\cos\theta = \frac{\sigma_{xx}}{2}\sin 2\theta$$



What And Where Are The Maximum Stresses !!??

For $(\sigma_\theta)_{max} \equiv \left[\frac{d\sigma_\theta}{d\theta}\right] = 0$

$$\sigma_\theta = \frac{\sigma_{xx}}{2}\left[1 + \cos 2\theta\right]$$

$$\frac{d\sigma_\theta}{d\theta} = \frac{\sigma_{xx}}{2}\left[0 + (-2\sin 2\theta)\right] = 0$$

$\rightarrow \sigma_{xx}\sin 2\theta = 0$

since $\sigma_{xx} \neq 0$

$\therefore \sin 2\theta = 0 \Rightarrow \therefore \theta = 0$

$\therefore (\sigma_\theta)_{max} = \frac{\sigma_{xx}}{2}\left[1 + \cos(\text{zero})\right] = \sigma_{xx}$

Thus the maximum direct stress (σ_θ) is given by σ_{xx} on a plane at right angle to the applied stress.

For maximum shear stress:

$$\frac{\partial \tau_\theta}{d\theta} = 0$$

$$\frac{d}{d\theta}\left[\frac{\sigma_{xx}}{2} \cdot \sin 2\theta\right] = \frac{2\sigma_{xx}\cos 2\theta}{2}$$

$\therefore \sigma_{xx}\cos 2\theta = 0$

since $\sigma_{xx} \neq 0$

$\therefore \cos 2\theta = 0$

$\therefore \theta = 45^\circ$

$$(\tau_\theta)_{max} = \frac{\sigma_{xx}\sin 90^\circ}{2} = \frac{1}{2}\sigma_{xx}$$, occurs at 45°.

Thus the maximum shear stress (τ_θ) is given by ($\frac{1}{2}\sigma_{xx}$) on a plane at 45°

$$\boxed{\begin{aligned}\sigma_\theta &= \sigma_{xx}\cos^2\theta \\ \tau_\theta &= \sigma_{xx}\sin\theta\cdot\cos\theta\end{aligned}}$$ Stress Transformation Equations

i.e. they transform the applied direct stress, σ_{xx} to stresses (σ_θ, τ_θ) on angle θ.

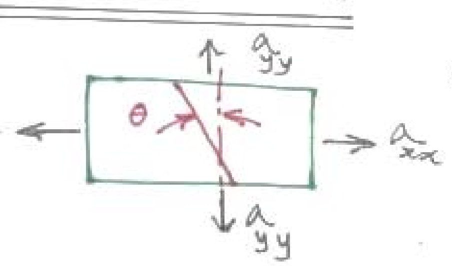
Sign Convention

σ_θ Tensile is +ve
Compressive is −ve

τ_θ Right direction to the outward normal +ve
Left " " " " " −ve

Element subjected to Two Direct stresses

Draw F.B.D. of part of the element, consider σ_{xx} force equilibrium, resolving forces to parallel and perpendicular to the plane.



Then go through similar procedures, to get the transformation equations:

$$\sigma_\theta = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta$$
$$= \frac{1}{2}(\sigma_{xx}+\sigma_{yy}) + \frac{1}{2}(\sigma_{xx}-\sigma_{yy})\cos 2\theta$$

And

$$\tau_\theta = \sigma_{xx}\cos\theta\sin\theta - \sigma_{yy}\cos\theta\sin\theta$$
$$= \frac{1}{2}(\sigma_{xx}-\sigma_{yy})\sin 2\theta$$

Element Subjected to Shear Stress Only

Using the same Procedures, gives:

$$\sigma_\theta = \tau_{xy}\cos\theta \cdot \sin\theta + \tau_{yx}\cos\theta \cdot \sin\theta$$

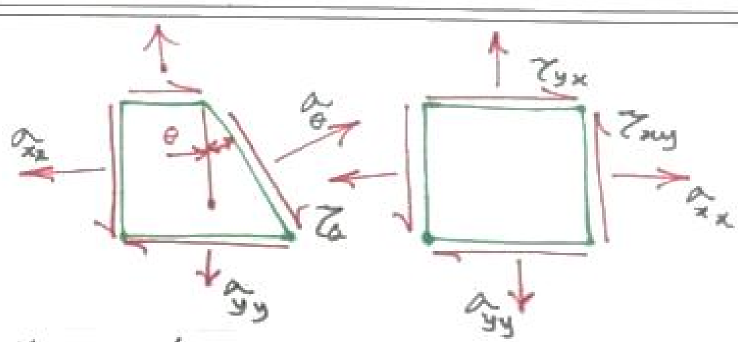
$$= 2\tau_{xy}\cos\theta \cdot \sin\theta \qquad (\tau_{xy} = \tau_{yx})$$

$$= \tau_{xy}\sin 2\theta$$

&

$$\tau_\theta = -\tau_{xy}\cos 2\theta$$

Element subjected To a General 2-Dimensions Stress System (Plane Stress)



This still could generate stress in the 3rd direction within the material. Combining the previous cases (Principle of superposition theory) gives:

$$\sigma_\theta = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_\theta = \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\sin 2\theta - \tau_{xy}\cos 2\theta$$

Rewriting the above plane stress transformation equations:

$$\sigma_\theta - \frac{1}{2}(\sigma_{xx}+\sigma_{yy}) = \frac{1}{2}(\sigma_{xx}-\sigma_{yy})\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_\theta = \frac{1}{2}(\sigma_{xx}-\sigma_{yy})\sin 2\theta - \tau_{xy}\cos 2\theta$$

Square both sides and add equations

$$\left[\sigma_\theta - \frac{1}{2}(\sigma_{xx}+\sigma_{yy})\right]^2 + \tau_\theta^2 = \frac{1}{4}(\sigma_{xx}-\sigma_{yy})^2 + \tau_{xy}^2$$

Compare the above equation with the equation of the circle, it's centre coordinates are (a, b) and its radius is (r):

$$(x-a)^2 + (y-b)^2 = r^2$$

The plane stress field equation represents a circle equation, where:

$$a = \frac{1}{2}(\sigma_{xx}+\sigma_{yy})$$

$$b = 0$$

And

$$r = \sqrt{\frac{1}{4}(\sigma_{xx}-\sigma_{yy})^2 + \tau_{xy}^2}$$

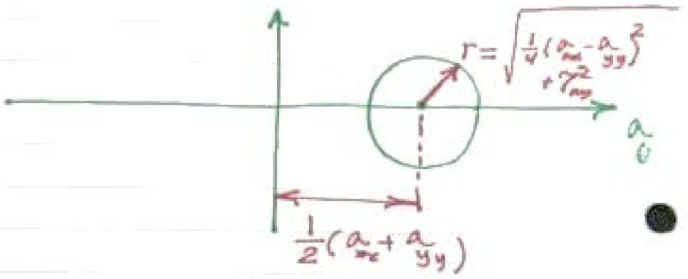
This circle have the following Conditions

1- This circle represents all possible states of direct and shear stress. Combinations on any plane through a stressed point in the material

2- plot direct stress as a horizontal axis.

3- Plot shear stress as a vertical axis.

4- The circle centre always on the horizontal axis (i.e $b=0$)

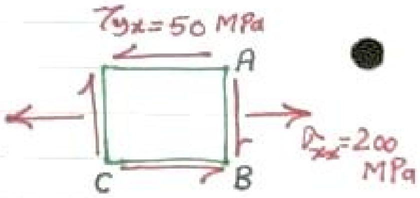


5- The angle on the element is doubled on the circle.

This circle is called MOHR'S CIRCLE represents the German engineer OTTO MOHR.

Quick Method for Drawing (Mohr's Circle)

1- Draw the σ axis and the τ axis.



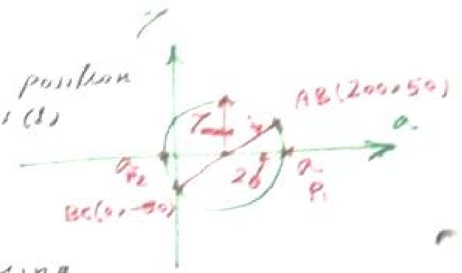
2- choose two neighbours sides of the element AB & BC. Perbendicular to each other.

3- The coordinates that represent any side is represented by its direct stress and shear stress. Each side on the element is a point on the circumfrence of Mohr's circle.

4- Use a proper scale to plot these two points (AB, BC).

5- Join the two points AB and BC by a straight line. Its midpoint represents the circle centre. And this point represents the intersection point with the (σ) axis in the case of $\tau_{xy} \neq 0$.

6- Now after locating the position of the principal stress(s) σ_{p_1} on Mohr's circle, its value could be counted by inverting this point coordinate using the scale value. Also its position (angle) on the element could be found by rotating the line (AB) about any point fall on it in the same direction as in the circle (clockwise ↻) with half value on the element (i.e θ). And σ_{p_2} in the element is perpendicular to σ_{p_1} direction



7- Also it is easy to count τ_{max} from Mohr's circle. And it's position on the element is in a right angle with (σ_{p_1}) face counterclockwise direction as shown.