Tikrit University
College of Engineering
Mechanical Engineering department

Third Class

Numerical Analysis Course

Dr. Ibrahim Thamer Nazzal

2018/2019

Numerical Analysis

WHAT IS NUMERICAL ANALYSIS?

The study of quantitative approximations to the solutions of mathematical problems including consideration of and bounds to the errors involved

TWO ISSUES OF NUMERICAL ANALYSIS:

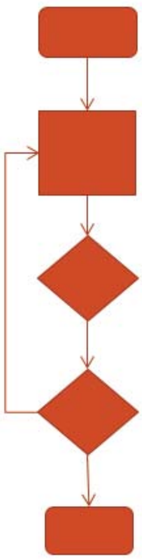
– How to compute? This corresponds to algorithmic aspects;
– How accurate is it? That corresponds to error analysis aspects

Reasons to study the numerical analysis

- **Non-linear equations**
- **Complex behaviours**
- **Analytical methods may not exist to solve for the exact roots or the exact solution**
- **Use of computers**
- **Flexibility of making changes**
- **Powerful software packages are available (special or general purpose).**

• IMPORTANT NOTES:

- Numerical analysis solution is always numerical.
- Results from numerical analysis is an approximation.



- Numerical methods follow the procedure
  - Step1: Initialize: Select some initial value
  - Step2: Estimate using (guess, some analytical technique) a new value at increment "i"
  - Step 3: Is the system converging? If not, use something else. We usually know a priori whether a method will converge or not form mathematics. Therefore, this step is often omitted.
  - Step 4: Is the change from the previous value to current value smaller than our acceptable error?
    - If not, make the current value the previous value and return to step 2.
    - If so, stop and accept the new value as the solution.

Nature of numerical problems

The field of numerical analysis explores the techniques that give approximate solutions to such problems with the desired accuracy.

NUMERICAL ERRORS

When we get into the real world from an ideal world and finite to infinite, errors arise.

SOURCES OF ERRORS:

• Mathematical problems involving quantities of infinite precision.

• Numerical methods bridge the precision gap by putting errors under firm control.

• Computer can only handle quantities of finite precision.

TYPES OF ERRORS:

1. ***Truncation errors*** are those errors corresponding to the fact that a finite (or infinite) sequence of computational steps necessary to produce an exact result is "truncated" prematurely after a certain number of steps.

• Truncation error (finite speed and time) - An example:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right) + \sum_{n=4}^{\infty} \frac{x^n}{n!}$$

$$= p_3(x) + \sum_{n=4}^{\infty} \frac{x^n}{n!}$$

2. ***Round of errors*** are errors arising from the process of rounding off during computation. These are also called *chopping*, i.e. discarding all decimals from some decimals on.

• Round-off error (finite word length): All computing devices represent numbers with some imprecision, except for integers.

3. ***Inherent errors or experimental errors***

• Human errors: (a) Mathematical equation/model. (b) Computing tools/machines. (c) Error in original data. (d) Propagated error.

Measure of errors

Let α be a scalar to be computed and let $\bar{\alpha}$ be its approximation.

Then, we define

- Absolute error = | true value – approximated value |.

$$\varepsilon = \left| \alpha - \bar{\alpha} \right|$$

- Relative error = $\left| \dfrac{\text{true value} - \text{approximated value}}{\text{true value}} \right|$

$$\varepsilon_r = \left| \frac{\alpha - \bar{\alpha}}{\alpha} \right|$$

True value = Approximate value + Correction.

Example, if $\tilde{a} = 10.52$ is an approximation to $a = 10.5$, then the error is $\varepsilon = 0.02$.

Example: Let the true value of p be 3.1415926535898 and its approximation be 3.14 as usual. Compute the absolute error and relative error of such an approximation.

The absolute error:

$$\varepsilon = \left| \pi - \bar{\pi} \right| = \left| 3.1415926535898 - 3.14 \right| = 0.0015926535898$$

which implies that the approximation is accurate up to 2 decimal places.

The relative error:

$$\varepsilon_r = \left| \frac{\pi - \bar{\pi}}{\pi} \right| = \frac{0.0015926535898}{3.1415926535898} = 0.000506957382897$$

STABILITY AND CONVERGENCE

STABILITY in numerical analysis refers to the trend of error change iterative scheme. It is related to the concept of convergence.

It is stable if initial errors or small errors at any time remain small when iteration progresses. It is unstable if initial errors or small errors at any time get larger and larger, or eventually get unbounded.

CONVERGENCE: There are two different meanings of convergence in numerical analysis:

a. If the discretized interval is getting finer and finer after dicretizing the continuous problems, the solution is convergent to the true solution.
b. For an iterative scheme, convergence means the iteration will get closer to the true solution when it progresses.

Solving Nonlinear Equations

If we have
$f(x)=0$
means to find such points that $f(r)=0$.
We call such point **roots of function** $f(x)$. A number $\boldsymbol{r}$ that satisfies an equation is called a root of the equation.

$f(x) = 0$ is called an **algebraic equation** if the corresponding $f(x)$ is **a polynomial.**
An example is $7x^2 + x - 8 = 0$.
$f(x) = 0$ is called **transcendental equation** if the $f(x)$ contains **trigonometric, or exponential or logarithmic** functions.
Examples of transcendental equations are $\sin x - x = 0$, $\tan x - x = 0$ and $7x^3 + \log(3x - 6) + 3e^x \cos x + \tan x = 0$.
How find the values **roots** of the equation $f(x) = 0$, or the **zeroes** of the function f (x).

Given a continuous function $f(x)$, find the value r such that $f(r) = 0$

$$\text{The equation: } x^4 - 3x^3 - 7x^2 + 15x = -18$$

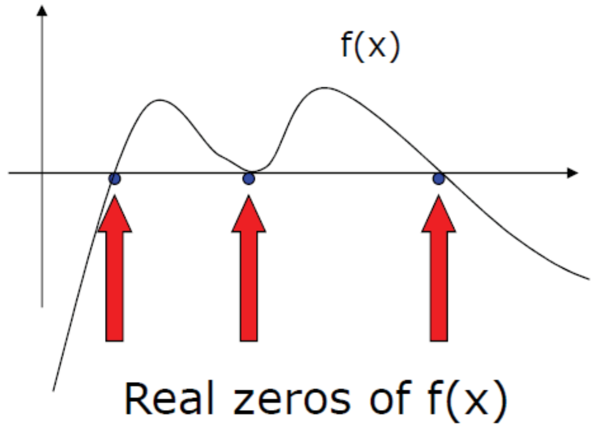
$$\text{has four roots: } -2,\ 3,\ 3, \textit{ and } -1\ .$$

$$\text{i.e., } x^4 - 3x^3 - 7x^2 + 15x + 18 = (x+2)(x-3)^2(x+1)$$

The equation has two simple roots (-1 and -3) and a repeated root (3) with multiplicity = 2.

Graphical Interpretation of Zeros

The real zeros of a function ***f*(*x*)** are the values of ***x*** at which the graph of the function crosses (or touches) the x-axis.



There are methods to find the roots of algebraic and transcendental equations of the form $f(x) = 0$.

Solution Methods

Several ways to solve nonlinear equations are possible:

1. Analytical Methods

Analytical Solutions are available for special equations only.

$$\text{Analytical solution of: } a\,x^2 + b\,x + c = 0$$

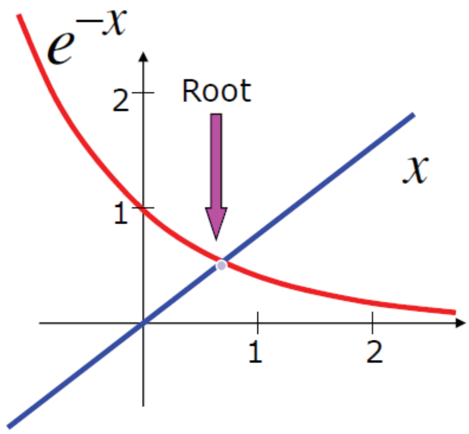
$$roots = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

No analytical solution is available for : $x - e^{-x} = 0$

2. Graphical Methods

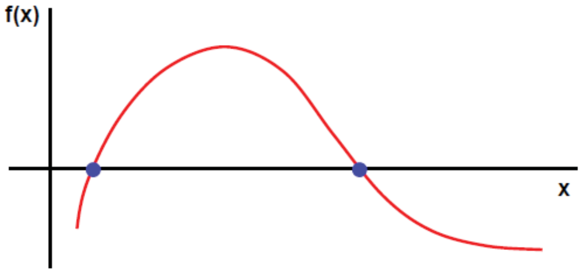
Graphical methods are useful to provide an initial guess to be used by other methods.

$$Solve$$
$$x = e^{-x}$$
$$The\ \ root \in [0,1]$$
$$root \approx 0.6$$



3. <u>Numerical Solutions</u>

Finding Roots of Equations using numerical solutions



f(x) is given

$f(x_r) = 0 \quad \rightarrow \quad x_r = ?$

Numerical methods of finding roots of zeros function are

Bracketing Methods

•Bisection Method

•False-Position Method

Open Methods

• Fixed point Iteration

•Newton-Raphson Method (Needs the derivative of the function.)
•Secant Method

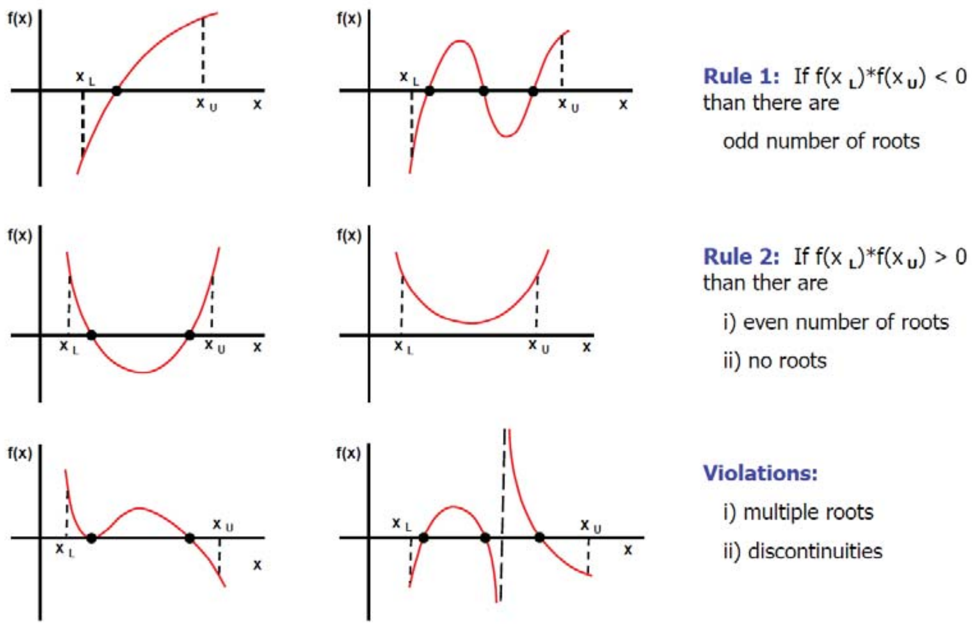
Bracketing Methods

In bracketing methods, the method starts with an **interval** that contains the root and a procedure is used to obtain a **smaller interval containing the root**.

Examples of bracketing methods:

- **Bisection method**
- **False position method**

General Idea of Bracketing Methods



Open Methods

- In the open methods, the method starts with **one or more initial guess points**. In each iteration, a new guess of the root is obtained.

- Open methods are usually more **efficient than bracketing methods**.

Fixed point Iteration Method

We have seen the non-linear equation is $f(x) = 0$

The idea of the fixed point iteration methods is to first reformulate a equation to an equivalent fixed point problem:

$$f(x) = 0 \quad \leftrightarrow \quad x = \phi(x)$$

and then to use the iteration: with an initial guess x_0 chosen, compute a sequence

$$x_{n+1} = \phi(x_n), \qquad n \geq 0$$

in the hope that $x_n \to \alpha$.

There are infinite many ways to introduce an equivalent fixed point problem for a given equation.

Now in this method the process is

- **To solve $f(x) = 0$.**
- **Rearrange $f(x)$ in such a way that $x = \phi(x)$**

→Provide initial guess for x say x_i

→Evaluate $g(x_i)$

→If not equal then, $x_{i+1} = g(x_i)$

→Evaluate $g(x_{i+1})$

→Continue till some tolerance ε i.e., $|x_{i+1} - x_i| \leq \varepsilon$

Example Solve $f(x) = x^2 + 3x + 1 = 0$, by fixed-point iteration method.

Solution

Write the given equation as

$x^2 = 3x - 1$ or $x = 3 - 1/x$.

Choose $g(x) = 3 - 1/x$

Then $\phi'(x) = \frac{1}{x^2}$ and $|\phi'(x) <| 1$ on the interval $(1, 2)$.

The iterative formula is given by

$$x_{n+1} = 3 - \frac{1}{x_n} \qquad (n = 0, 1, 2, \ldots)$$

Starting with, $x_0 = 1$, we obtain the sequence

$$x_0 = 1.000,\ x_1 = 2.000,\ x_2 = 2.500,\ x_3 = 2.600,\ x_4 = 2.615, \ldots$$

Example **Find a real root of the equation $x^3 + x^2 - 1 = 0$ on the interval [0, 1] with an accuracy of 10^{-4}.**

To find this root, we rewrite the given equation in the form

$$x = \frac{1}{\sqrt{x+1}}$$

Take

$$\phi(x) = \frac{1}{\sqrt{x+1}}. \text{ Then } \phi(x) = -\frac{1}{2}\frac{1}{(x+1)^{\frac{3}{2}}}$$

Hence the iteration method gives:

n	x_n	$\sqrt{x_n+1}$	$x_{n+1}=1/\sqrt{x_n+1}$
0	0.75	1.3228756	0.7559289
1	0.7559289	1.3251146	0.7546517
2	0.7546617	1.3246326	0.7549263

At this stage,

$$|x_{n+1}-x_n| = 0.7549263 - 0.7546517 = 0.0002746,$$

which is less than 0.0004. The iteration is therefore terminated and the root to the required accuracy is 0.7549

***Example :* Use the method of iteration to find a positive root, between 0 and 1, of the equation $x\,e^x = 1$.**

Writing the equation in the form

$x = e^{-x}$.

We find that $\phi(x) = e^{-x}$ and so $\phi'(x) = -e^{-x}$.

Hence $|\phi'(x)| < 1$ for $x < 1$, which assures that the iterative process defined by the equation $x_{n+1} = \phi(x_n)$ will be convergent, when $x < 1$.

The iterative formula is

$$x_{n+1} = \frac{1}{e^{x_n}} \qquad (n = 0, 1, \dots)$$

Starting with $x_0 = 1$, we find that the successive iterates are given by

$$x_1 = 1/e = 0.3678794, \quad x_2 = \frac{1}{ex_1} = 0.6922006,$$

$$x_3 = 0.5004735, \qquad x_4 = 0.6062435,$$

$$x_5 = 0.5453957, \qquad x_6 = 0.5796123,$$

We accept 6.5453957 as an approximate root.

***Example* Find the root of the equation $2x$ = cos x + 3 and correct to three decimal places.**

We rewrite the equation in the form