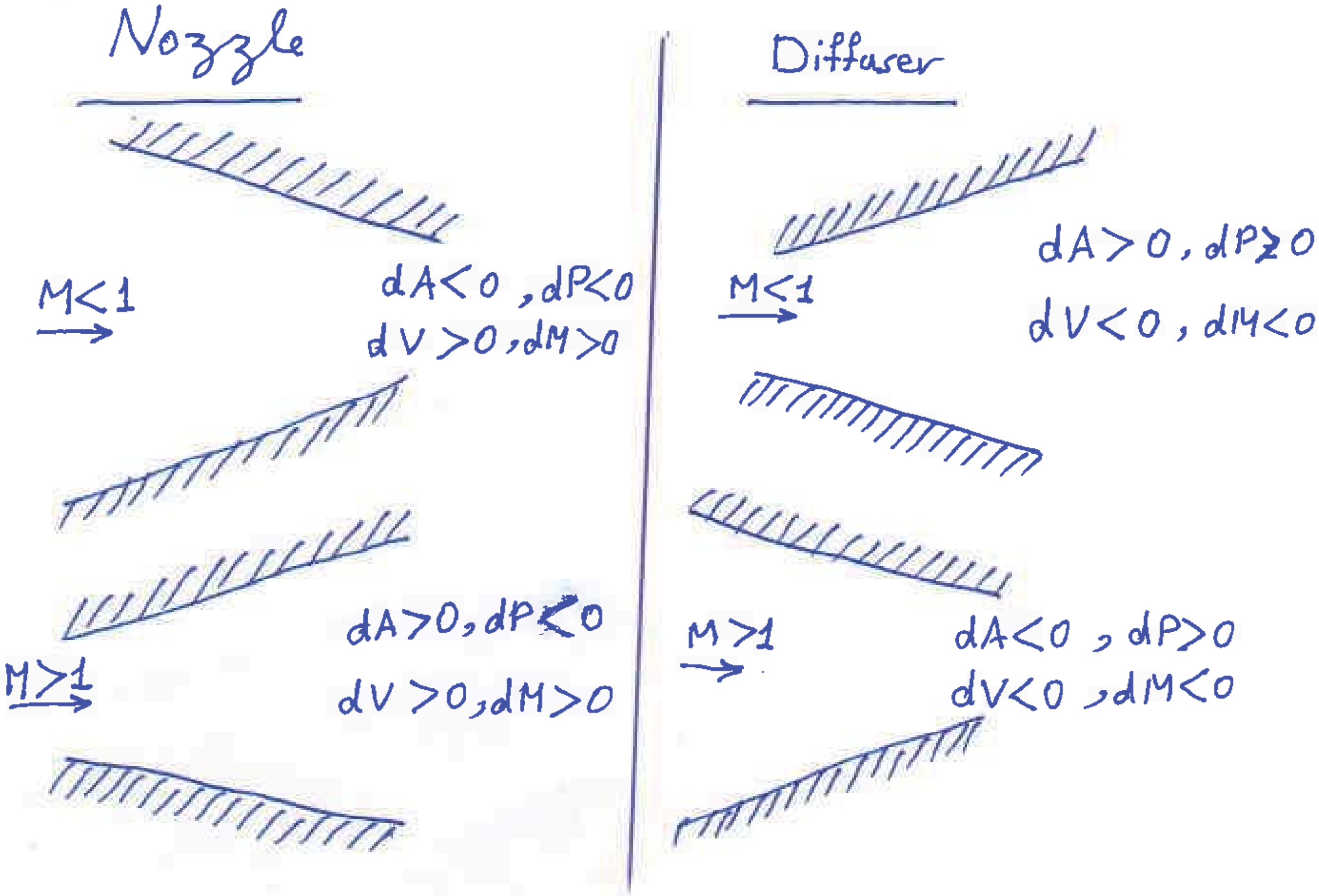
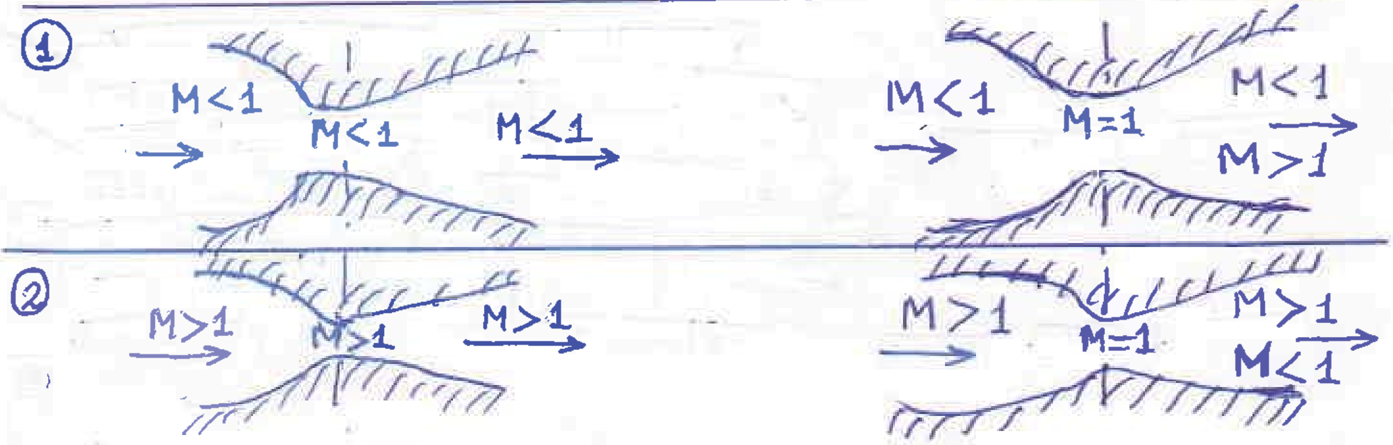


It can be concluded from the above equations that $M = 1$ at $dA = 0$.

. The variation of pressure with cross sectional area can be written as follows:

$$\frac{dP}{dx} = \frac{\rho V^2}{1 - M^2} * \frac{1}{A} \cdot \frac{dA}{dx}$$

3.3. Mach Number Possibility at The Throat



1- $M < 1$ at inlet

a- If $M<1$ at the throat, $M<1$ at outlet

b- If $M=1$ at the throat, there are two possible cases based on flow conditions at exit; the first case is $M<1$ and the second case is $M>1$.

2- $M>1$ at inlet

a- If $M>1$ at the throat, $M>1$ at outlet.

b- If $M=1$ at the throat, there are two possible cases based on flow conditions at exit; the first case is $M>1$ and the second case is $M<1$.

3.4. Critical Conditions

The flow conditions at the throat when the velocity reaches $M=1$ is called to be at critical condition.

$$\frac{T_o}{T} = 1 + \frac{\gamma-1}{2} M^2$$

Substituting $M=1$, the above equation will be

$$\frac{T^*}{T_o} = \frac{2}{\gamma+1} = 0.835 \quad (\text{If } \gamma = 1.4) \text{ —— } (3.6)$$

By same way

$$\frac{P^*}{P_o} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} = 0.52828 \quad (\text{If } \gamma = 1.4) \text{ —— } (3.7)$$

and

$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} = 0.63394 \quad (\text{If } \gamma = 1.4) \text{ —— } (3.8)$$

If $\frac{T}{T_o} > 0.835 \Rightarrow M < 1$

If $\frac{T}{T_o} < 0.835 \Rightarrow M > 1$

If $\frac{P}{P_o} > 0.52828 \Rightarrow M < 1$

If $\frac{P}{P_o} < 0.52828 \Rightarrow M > 1$

If $\frac{\rho}{\rho_o} > 0.63394 \Rightarrow M < 1$

If $\frac{\rho}{\rho_o} < 0.63394 \Rightarrow M > 1$

3.5. Isentropic Flow Equations

Stagnation state is constant in isentropic flow, therefore, it represents a suitable reference for variables P, T & ρ. So the relations for those variables (P, T & ρ) between stagnation point and any other point in the duct are given as

$$\frac{T_o}{T} = 1 + \frac{\gamma-1}{2} M^2 \qquad (3.9)$$

$$\frac{P_o}{P} = \left[1 + \frac{\gamma-1}{2} M^2\right]^{\frac{\gamma}{\gamma-1}} \qquad (3.10)$$

$$\frac{\rho_o}{\rho} = \left[1 + \frac{\gamma-1}{2} M^2\right]^{\frac{1}{\gamma-1}} \qquad (3.11)$$

However, stagnation state is not suitable reference state for the cross section area of duct (A), gas velocity (V) and impulse function (F). This is because stagnation state may represents a larg reservoir ($A \simeq \infty$) having stagnant gas ($V \simeq 0$) and supplying gas to the duct. Therefor, the critical state is the suitable reference state for the variables A, V & F and as follows.

$$M^* = \frac{V}{V^*} = \left[\frac{\frac{\gamma+1}{2} M^2}{1 + \frac{\gamma-1}{2} M^2} \right]^{\frac{1}{2}} \quad (3.12)$$

Note: where M^* doesn't mean M when $(M=1)$.

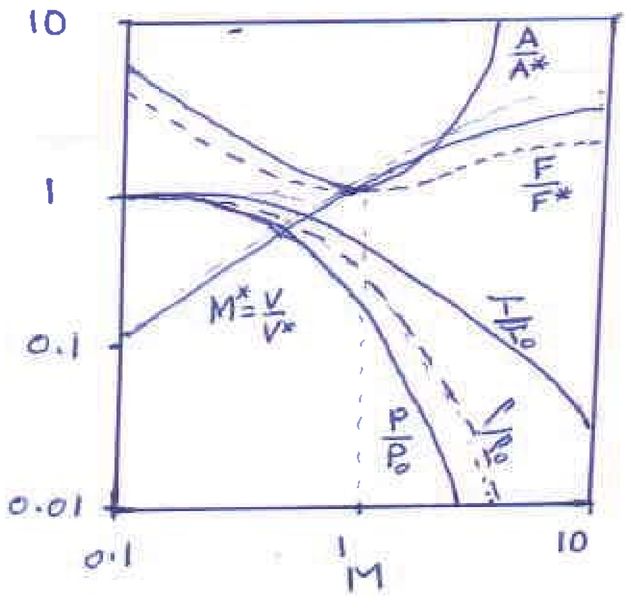
$$\frac{F}{F^*} = \frac{1+\gamma M^2}{M\sqrt{2(\gamma+1)\left(1+\frac{\gamma-1}{2}M^2\right)}} \quad (3.13)$$

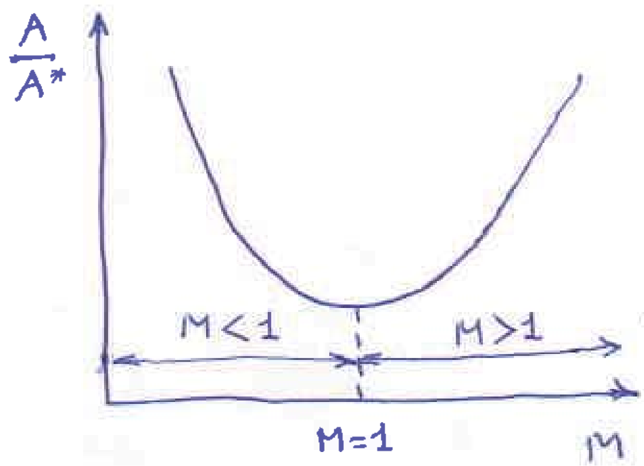
$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1+\frac{\gamma-1}{2}M^2}{\frac{\gamma+1}{2}} \right]^{\frac{\gamma+1}{2(\gamma-1)}} \quad (3.14)$$

(Derive this very important Equation)

Equations (3.9) to (3.14) are tabulated in Table (C-1) as a functions of Mach number (M) for gases of $\gamma = 1.4$.

Also, those functions can be shown in this figure.





3.6. Choking ظاهرة الإختناق

The relation between mass flow rate to area unit $\left(\frac{\dot{m}}{A}\right)$ and Mach number can be found as follows:

$$\dot{m} = \rho V A \qquad \text{(Continuity Eq.)}$$

$$\frac{\dot{m}}{A} = \frac{P}{RT} M \sqrt{\gamma R T} * \frac{P_o}{P_o} * \frac{\sqrt{T_o}}{\sqrt{T_o}} = \sqrt{\frac{\gamma}{R}}\, M \frac{P}{P_o} \cdot \sqrt{\frac{T_o}{T}} \cdot \frac{P_o}{\sqrt{T_o}}$$

By substituting Eqs (3.9) & (3.10) in the above equ, the following equ. is obtained

$$\boxed{\frac{\dot{m}\sqrt{T_o}}{P_o\, A} = \sqrt{\frac{\gamma}{R}}\; \frac{M}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}}} \qquad \text{———— (3.15)}$$

١٤.١١.١٨

From equ. (3.15) in which $\frac{\dot{m}\sqrt{T_o}}{P_o\, A}$ is the mass flow parameter,

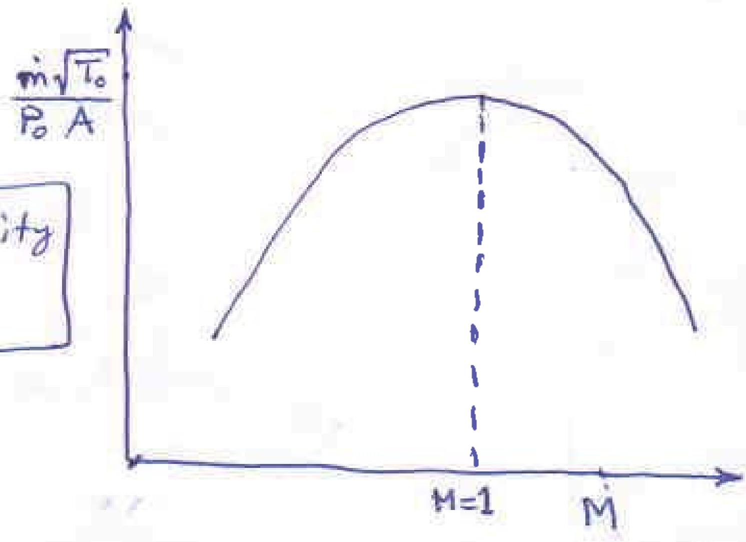
$$\dot{m} = \dot{m}_{max} \quad \text{when} \quad M = 1 \Rightarrow A = A^*$$

Hence,

$$\frac{\dot{m}_{max}\sqrt{T_o}}{P_o\, A^*} = \left(\frac{\dot{m}\sqrt{T_o}}{P_o\, A^*}\right)_{max} \quad \text{when} \quad M = 1 \left(\text{or} \ \frac{P}{P_o} = \frac{P^*}{P_o}\right) \text{— (3.15a)}$$

$$\therefore \left(\frac{\dot{m}\sqrt{T_o}}{P_o\, A^*}\right)_{max} = \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} \qquad \text{———— (3.16)}$$

The fact (3.15a) can be shown in this figure.

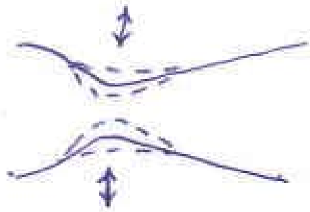


> **Note:** M cannot be equal to unity only at throat.

In other word, $\frac{\dot{m}}{A_t}$ has maximum value when $M_t = 1$ (subscript t refers to throat, A_t: Throat area, M_t: Mach number at the throat). This phenomenon is called "Choking".

Note: ① A_t has minimum value if $\dot{m}$ is fixed (certain value).
② $\dot{m}$ has maximum value if A_t is fixed (certain value).

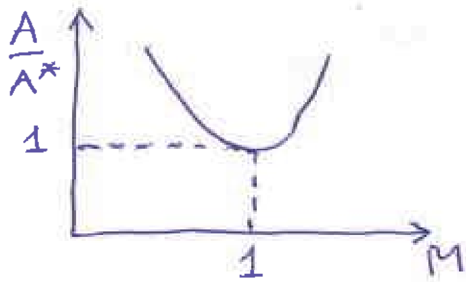
There are two cases to be considered:



Case 1: A_t is variable & $\dot{m}$ is fixed

- If A_t decreases $\Rightarrow \frac{\dot{m}}{A_t}$ increases
- $\frac{\dot{m}}{A_t}$ reaches its maximum value $\left(\left(\frac{\dot{m}}{A_t}\right)_{max}\right)$ when A_t is reduced to its minimum value (A_{min}) at which $M_t = 1$ & $P_t = P^*$ ($A_{min} = A^*$).
- A_t cannot be less than A_{min} (or A^*), why?

Ans① Mathematically:



- Based on this figure (eq. 3.14), there is no solution if $A < A^*$ (there is no M for $\frac{A}{A^*} < 1$). In conclusion, the only one way to reduce A_t is to reduce $\dot{m}$ somehow so $\frac{\dot{m}}{A_t}$ maintains its maximum value and $M_t = 1$ and hence $(A_t)_{new}$ will be A_{min}.

Ans② Practically:
If A_t is reduced to less than A_{min}, $\dot{m}$ will be reduced and flow characteristics will be changed accordingly at each point inside the duct.

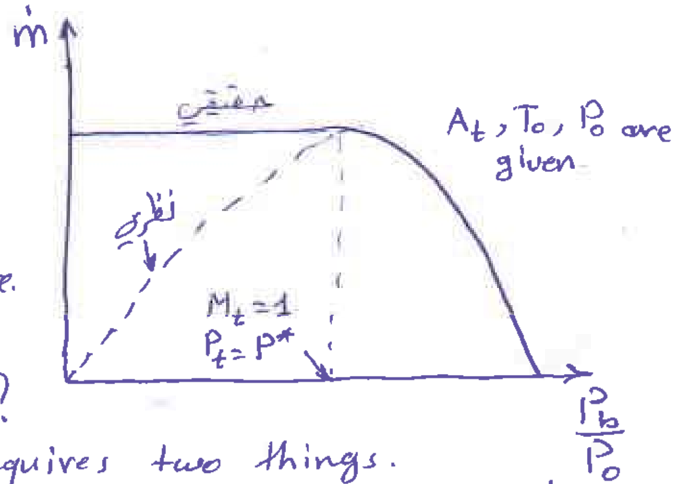
Case 2 : A_t is fixed & $\dot{m}$ is variable

If $\dot{m}$ increases by reducing back pressure (P_b) $\Rightarrow \frac{\dot{m}}{A_t}$ increases until $\frac{\dot{m}}{A_t} = \left(\frac{\dot{m}}{A_t}\right)_{max}$ (i.e. $\dot{m} = \dot{m}_{max}$) at which $M_t = 1$ & $P_t = P^*$.

Eq. 3.15 can be written as follows.

$$\frac{\dot{m}\sqrt{T_o}}{P_o\, A} = \sqrt{\frac{2\gamma}{R(\gamma-1)}} \left(\frac{P}{P_o}\right)^{\frac{1}{\gamma}} \sqrt{1-\left(\frac{P}{P_o}\right)^{\frac{\gamma-1}{\gamma}}} \quad \text{———} (3.17)$$

Note : What happens to $\dot{m}$ if P_b continues reducing?



Theoretically: Based on eqn. 3.17, $\dot{m}$ decreases as shown in this figure.

Practically: Remains constant at its maximum value ($\dot{m}_{max}$), why?

Any change in $\dot{m}$ inside the duct requires two things. Firstly, P_b should changes and secondly, it needs to send information about the change of P_b to the inlet of the duct. As the information about the pressure moves at speed of sound, information wave dose not reach the inlet passing from outlet through the throat of the duct when $M_t = 1$, i.e., the resultant of velocity equals to zero at throat. So, $\dot{m}$ remains constant at its maximum value as shown in the figure above.

Summary: Choking occurs in subsonic/supersonic flow when $M_t = 1$ and $\frac{\dot{m}}{A} = \left(\frac{\dot{m}}{A}\right)_{max}$ either by $\dot{m}_{max}$ passes through a certain A_t or by $(A_t)_{min}$ for a certain $\dot{m}$.

For ease of calculations and by substituting R & γ of air, eqn 3.16 will be:

$$\frac{\dot{m}\sqrt{T_o}}{P_o\, A} = 0.04044 \quad s/m.K^{1/2} \quad \text{———} (3.18)$$

3.7. Isentropic Flow in a Converging Nozzle

Flow in converging nozzle shown in figure (3.10) is supplied from a large chamber, where the stagnation conditions are assumed. Flow is induced by a vacuum pump downstream and is controlled by the valve shown.

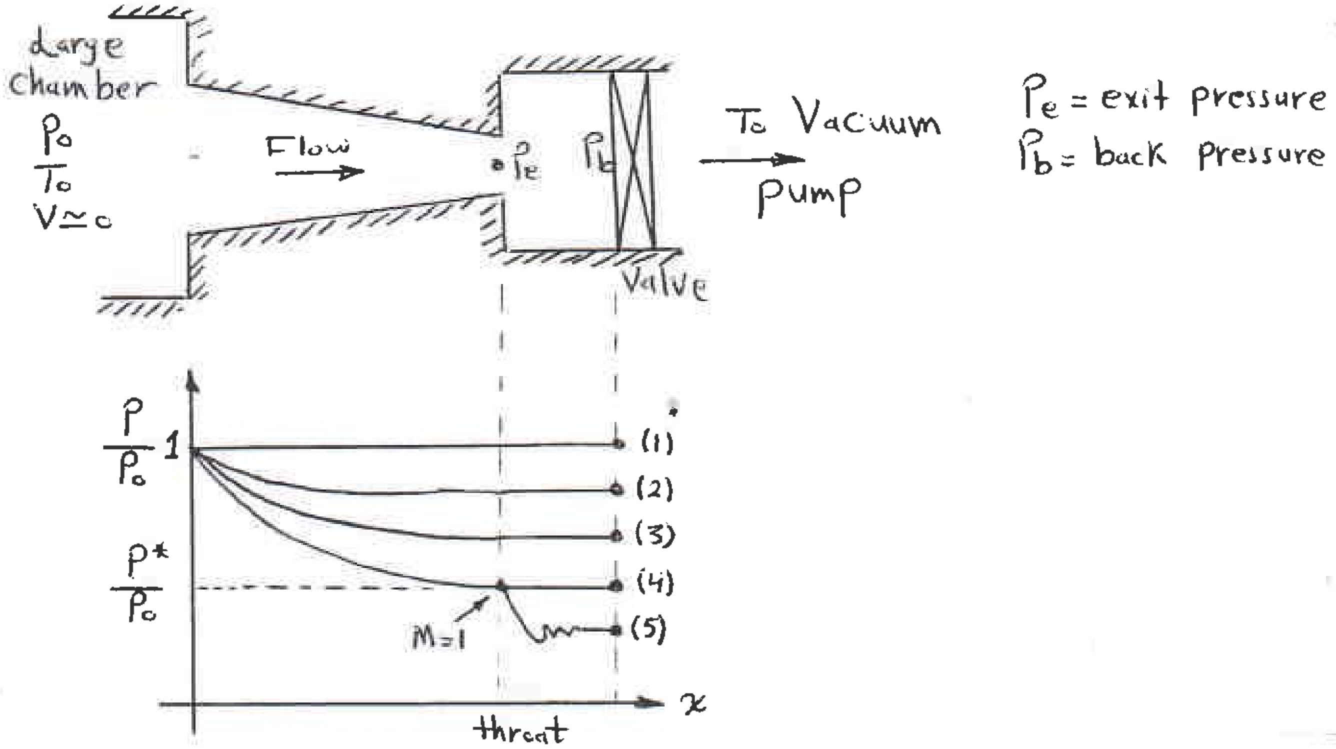


Figure (3.10) Converging nozzle operating at various back pressures.

① When the valve is closed. $\dot{m} = 0$, $P_o = P_e = P_b$ **Condition (1) Figure (3.10)**

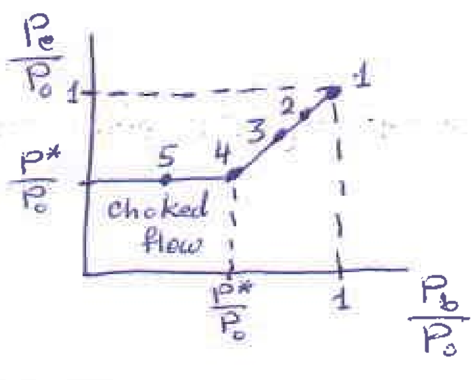
② If the back pressure (P_b) is reduced to slightly less than (P_o), there will be flow through the nozzle with a decrease in pressure in the direction of flow as shown by condition (2). Flow at exit is subsonic ($P_e = P_b$)

③ As we continue to decrease (P_b), ($\dot{m}$) will continue to increase and (P_e) continue to decrease as shown by condition (3).

④ with continue to decrease (P_b) → $M_e = 1$ and $P_e = P^*$ ($\dot{m} = \dot{m}_{max}$) [choked nozzle]

The critical pressure ratio for ideal gas can be obtained from equation (3.7)

⑤ If P_b is reduced below P^*, there is no effect neither on the pressure distribution through the nozzle, P_e nor $\dot{m}$ as discussed previously in section 3.6.



3.8. Isentropic Flow in Converging - Diverging Nozzle

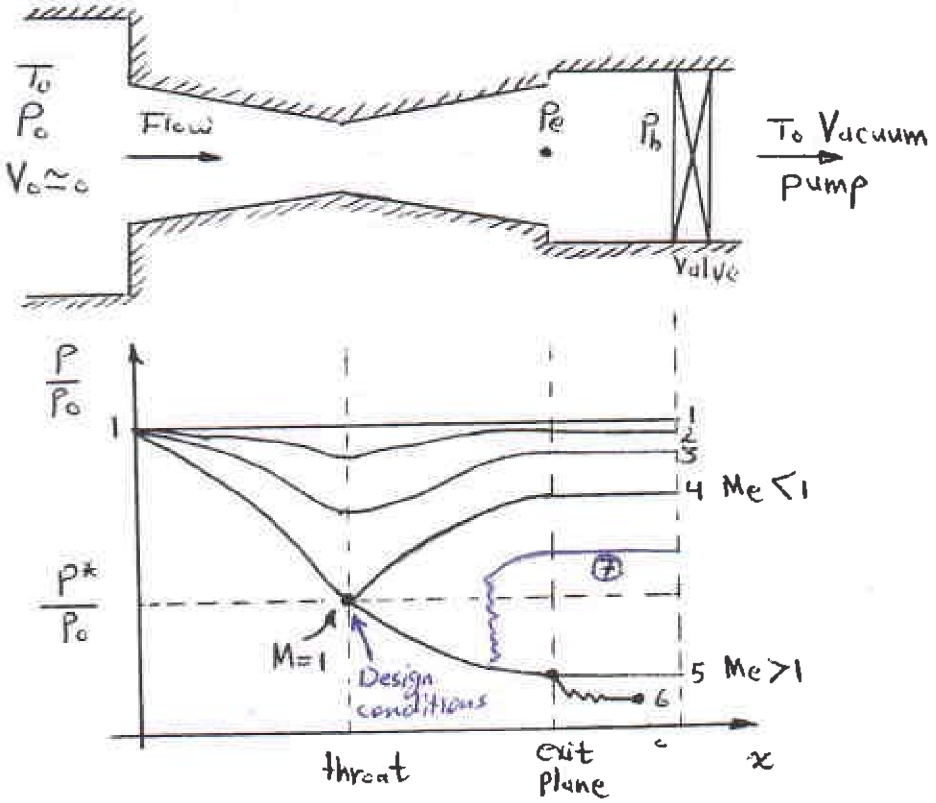


Figure (3.12) Converging - Diverging Nozzle operating at Various back Pressures.

① When the Valve is closed, $\dot{m} = 0$ $P_o = P_e = P_b$ Condition (1) Figure (3.12)

② Opening the Valve slightly produces Curve (2) figure (3.12) $\dot{m} \uparrow$

③ As the Valve is opened further produces curve (3) figure (3.12) $\dot{m} \uparrow\uparrow$

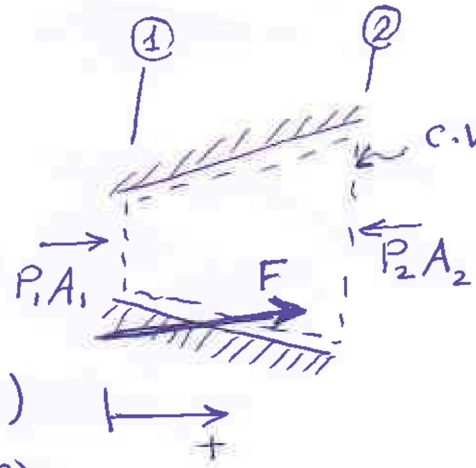
④ As the Valve is opened more, curve (4) results. M=1 at throat and the nozzle is choked (i.e. $\dot{m} = \dot{m}_{max}$ equation (3.16))

⑤ To accelerate flow in the diverging section requires pressure decrease. This Condition is illustrated by curve (5) in figure (3.12). A converging - diverging nozzle generally is used to produce supersonic flow at exit plane. If P_b is set at P_5, flow will be isentropic through the nozzle and supersonic at the exit. Nozzles operating at ($P_b = P_5$) are said to operate at design conditions.

⑥ If $P_b < P_5$ (curve 6) $\dot{m} = \dot{m}_{max}$ and flow isentropic from chamber to exit (same curve (5)), and then undergo nonisentropic expansion. (Underexpanded Nozzle)

⑦ $P_4 > P_b > P_5$ $\longrightarrow$ Shock appears in the diverging part (overexpanded Nozzle)

3.9. Impulse Function



Momentum Eq.

$$\Sigma F = ma$$

$$F + P_1A_1 - P_2A_2 = \dot{m}(V_2 - V_1)$$

$$F = (P_2A_2 + \dot{m}V_2) - (P_1A_1 + \dot{m}V_1)$$

$$= (P_2 + \rho_2 V_2^2)A_2 - (P_1 + \rho_1 V_1^2)A_1$$

$$\therefore F = F_2 - F_1$$

Generally,

$$F = A(P + \rho V^2)$$

$$F = PA\left(1 + \frac{\rho}{P}V^2\right) = PA\left(1 + \frac{1}{RT}\frac{\gamma}{\gamma}V^2\right)$$

$$F = PA(1 + \gamma M^2)$$

Note: If ① Area is constant
② No friction in the flow $\Big\} \Rightarrow F = 0$

So, the impulse ratio is given as

$$\boxed{\frac{F}{F^*} = \frac{1 + \gamma M^2}{M\sqrt{2(\gamma+1)\left(1 + \frac{\gamma-1}{2}M^2\right)}}}$$

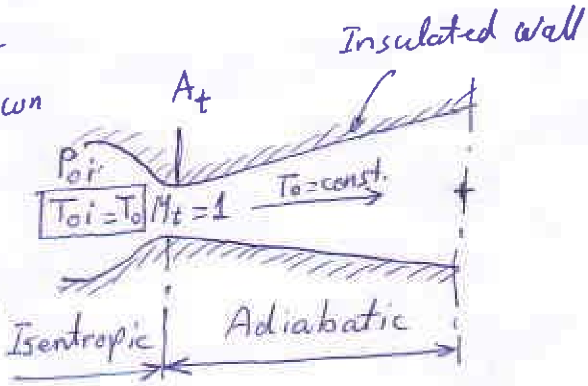
Important

Derive

which is same of Eq. 3.13 mentioned previously in section 3.5.

3.10. Important Notes on Isentropic Flow & Adiabatic Flow

In a general variable area adiabatic Flow (T_0=const.) as shown in the figure, the isentropic flow equations can be used at some point in that general flow. Hence, Eq. 3.16. shows that $P_0 A^*$ is constant, so



$$P_0 A^* = \text{constant in adiabatic Flow} = P_{0i} A_t \qquad (3.19)$$

$$\frac{A}{A_t} \cdot \frac{P}{P_{0i}} = \frac{A}{A^*} \cdot \frac{P}{P_0} = \frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}}{M\left(1+\frac{\gamma-1}{2}M^2\right)^{1/2}} \qquad (3.20)$$

This value is tabulated

Note: $\frac{A}{A_t} \neq \frac{A}{A^*}$, $\frac{P}{P_{0i}} \neq \frac{P}{P_0}$ (adiabatic flow)

Chapter -4-
Shock Waves

4.1. Introduction

It has been found experimentally, it is possible for a spontaneous change to occur in a flow, the velocity decreasing and the pressure increasing, this change is termed a shock wave.

shock wave can only occur if the initial flow is supersonic.

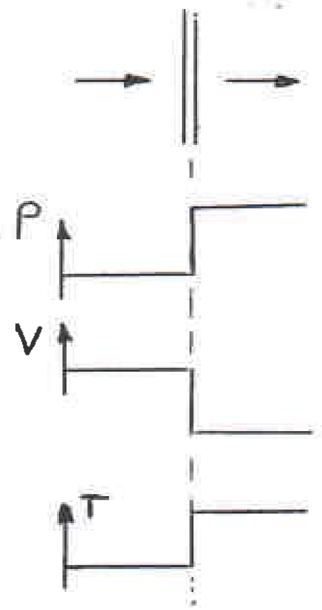
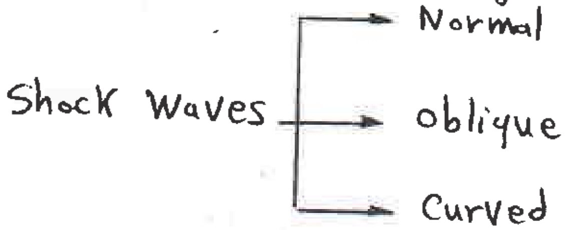


Figure (4.1) Changes through a Normal Shock Wave.



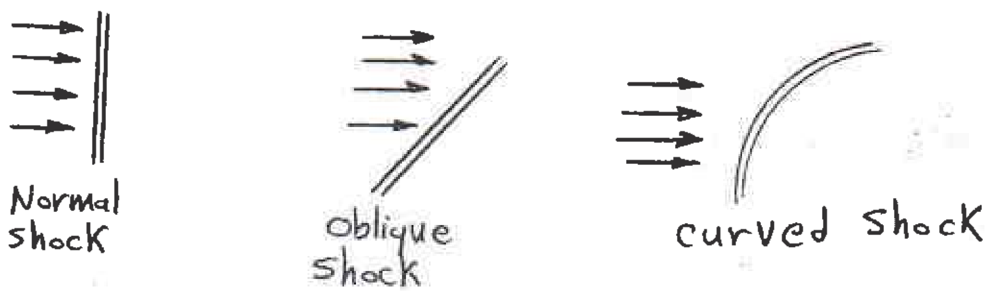


Figure (4.2) Shock Waves

4.2 Governing Equations of the Stationary Normal Shock Waves

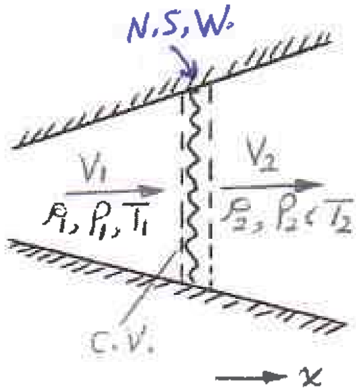


Figure (4-3)

Assumptions

1- The normal shock wave (N.S.W.) takes place at constant cross-sectional area. (Shock thickness is very small $\simeq 0.25 - 1\ \mu m$) $\Rightarrow A_1 = A_2$

2- Flow is adiabatic

Continuity Equation $\dot{m} = \rho V A$

$$\rho_1 V_1 \cancel{A_1} = \rho_2 V_2 \cancel{A_2}$$

$$\rho_1 V_1 = \rho_2 V_2 \text{ ----------- } (4.1)$$

Momentum Equation (x-momentum) $\sum F = \dot{m} V_2 - \dot{m} V_1$

$$P_1 \cancel{A_1} - P_2 \cancel{A_2} = \rho_2 \cancel{A_2} V_2^2 - \rho_1 \cancel{A_1} V_1^2$$

$$P_2 + \rho_2 V_2^2 = P_1 + \rho_1 V_1^2$$

Noting that the term ρV^2 in the momentum equation

$$\rho V^2 = \frac{P}{\cancel{RT}} M^2 \gamma \cancel{RT} = P \gamma M^2$$

$$\therefore P_2 + P_2 \gamma M_2^2 = P_1 + P_1 \gamma M_1^2 \xrightarrow{\text{or}} P_2 (1 + \gamma M_2^2) = P_1 (1 + \gamma M_1^2)$$

$$\boxed{\frac{P_2}{P_1} = \frac{(1 + \gamma M_1^2)}{(1 + \gamma M_2^2)}} \text{ ------- } (4.2)$$

Energy Equation

$$\cancel{q} = \left(h_2 + \frac{V_2^2}{2}\right) - \left(h_1 + \frac{V_1^2}{2}\right)$$

For ideal gas $h = C_p T$

$$\therefore C_p T_2 + \frac{V_2^2}{2} = C_p T_1 + \frac{V_1^2}{2}$$

But $C_p = \frac{\gamma R}{\gamma - 1}$ & $M = \frac{V}{\sqrt{\gamma R T}}$

$$\boxed{\frac{T_2}{T_1} = \frac{1 + \frac{\gamma - 1}{2} M_1^2}{1 + \frac{\gamma - 1}{2} M_2^2}} \text{ ------ } (4.3)$$

Density relation follows from the ideal gas equation

$$\frac{\rho_2}{\rho_1} = \frac{P_2}{\cancel{R} T_2} \frac{\cancel{R} T_1}{P_1} = \frac{P_2}{P_1} \frac{T_1}{T_2} \text{ ------ } (4.4)$$

$\because \rho V = \frac{P}{RT} M\sqrt{\gamma R T}$, equation (4.1) can be written as:-

$$\frac{P_1}{\sqrt{T_1}} M_1 = \frac{P_2}{\sqrt{T_2}} M_2 \longrightarrow \sqrt{\frac{T_2}{T_1}} = \frac{M_2}{M_1}\,\frac{P_2}{P_1}$$

Substitution of equs. (4.2) and (4.3) in the above equation gives:-

$$\left[\frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2}\right]^{1/2} = \left(\frac{M_2}{M_1}\right)\left(\frac{(1+\gamma M_1^2)}{(1+\gamma M_2^2)}\right)$$

Squaring, we obtain

$$\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2} = \left(\frac{M_2}{M_1}\right)^2\left[\frac{1+2\gamma M_1^2+\gamma^2 M_1^4}{\underbrace{1+2\gamma M_2^2+\gamma^2 M_2^4}}\right]$$

or

$$(2M_2^2+(\gamma-1)M_2^4)+(2M_2^2+(\gamma-1)M_2^4)(2\gamma M_1^2+\gamma^2 M_1^4) =$$
$$(2M_1^2+(\gamma-1)M_1^4)+(2M_1^2+(\gamma-1)M_1^4)(2\gamma M_2^2+\gamma^2 M_2^4)$$

or

$$(\gamma-1)(M_2^4-M_1^4)+2(M_2^2-M_1^2)+M_1^2M_2^2\left[((2+(\gamma-1)M_2^2)(2\gamma+\gamma^2M_1^2))-\right.$$
$$\left.((2+(\gamma-1)M_1^2)(2\gamma+\gamma^2M_2^2))\right]=0$$

or

$$(\gamma-1)(M_2^4-M_1^4)+2(M_2^2-M_1^2)+M_2^2M_1^2\left[(\cancel{4\gamma}+2\gamma^2M_1^2+2\gamma(\gamma-1)M_2^2+\gamma^2(\gamma-1)\cancel{M_1^2M_2^2})\right.$$
$$\left.-(\cancel{4\gamma}+2\gamma^2M_2^2+2\gamma(\gamma-1)M_1^2+\gamma^2(\gamma-1)\cancel{M_1^2M_2^2})\right]=0$$

or

$$(\gamma-1)(M_2^4-M_1^4)+2(M_2^2-M_1^2)+M_2^2M_1^2\left[(2\gamma^2\cancel{M_1^2}+2\gamma^2\cancel{M_2^2}-2\gamma M_2^2)\right.$$
$$\left.-(2\gamma^2\cancel{M_2^2}+2\gamma^2\cancel{M_1^2}-2\gamma M_1^2)\right]=0$$

$$\therefore (\gamma-1)(M_2^4-M_1^4)-2\gamma M_1^2M_2^2(M_2^2-M_1^2)+2(M_2^2-M_1^2)=0$$

or

$$(M_2^2-M_1^2)((\gamma-1)(M_2^2+M_1^2)-2\gamma M_1^2M_2^2+2)=0$$

Sol 1

$M_2^2 = M_1^2$ this means that there is no change in the Mach number across the shock wave, so that this solution is trivial.

Sol 2

$$(\gamma-1)(M_2^2+M_1^2) - 2\gamma M_1^2 M_2^2 + 2 = 0$$

this gives :-

$$\boxed{M_2^2 = \frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)}} \quad \text{-------} (4.5)$$

substitution of equ.(4.5) into equ.(4.3) gives :-

$$\frac{T_2}{T_1} = \frac{2+(\gamma-1)M_1^2}{\left(2+(\gamma-1)\left(\frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)}\right)\right)}$$

$$\boxed{\frac{T_2}{T_1} = \frac{(2+(\gamma-1)M_1^2)(2\gamma M_1^2-(\gamma-1))}{(\gamma+1)^2 M_1^2}} \quad \text{-------} (4.6)$$

And substitution of equ.(4.5) into equ.(4.2) gives :-

$$\frac{P_2}{P_1} = \frac{1+\gamma M_1^2}{\left[1+\gamma\left(\frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)}\right)\right]} = \frac{(1+\gamma M_1^2)(2\gamma M_1^2-(\gamma-1))}{(2\gamma M_1^2-(\gamma-1))+\gamma(\gamma-1)M_1^2+2\gamma}$$

$$= \frac{(1+\gamma M_1^2)(2\gamma M_1^2-(\gamma-1))}{2\gamma M_1^2-(\gamma-1)+\gamma^2 M_1^2-\gamma M_1^2+2\gamma} = \frac{(1+\gamma M_1^2)(2\gamma M_1^2-(\gamma-1))}{\underbrace{\gamma^2 M_1^2+\gamma M_1^2}_{(\gamma+1)M_1^2\gamma}+\gamma+1}$$

$$\frac{P_2}{P_1} = \frac{\cancel{(\gamma M_1^2+1)}(2\gamma M_1^2-(\gamma-1))}{(\gamma+1)\cancel{(\gamma M_1^2+1)}} = \boxed{\frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)}} \quad \text{------} (4.7)$$

substitution of equs. (4.6) and (4.7) into equ.(4-4) gives :-

$$\frac{\rho_2}{\rho_1} = \frac{\cancel{2\gamma M_1^2-(\gamma-1)}}{(\gamma+1)} \; \frac{(\gamma+1)^2 M_1^2}{(2+(\gamma-1)M_1^2)\cancel{(2\gamma M_1^2-(\gamma-1))}}$$

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}} \quad \text{------} (4.8)$$

The stagnation pressure ratio across the normal shock wave is obtained by:-

$$\frac{P_{o_2}}{P_{o_1}}=\frac{P_{o_2}}{P_2}\,\frac{P_1}{P_{o_1}}\,\frac{P_2}{P_1}=\frac{P_{o_2}/P_2}{P_{o_1}/P_1}\;\underbrace{\frac{P_2}{P_1}}_{\text{equ.}(4.7)}$$

$$\frac{P_{o_2}}{P_{o_1}}=\left[\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}}\frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)}$$

sub. (M_2^2) from equ.(4.5) in the above equation, we get :-

$$\frac{P_{o_2}}{P_{o_1}}=\left[\frac{1+\overbrace{\left(\frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)}\right)}^{M_2^2}\frac{\gamma-1}{2}}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}}\frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)}$$

$$\frac{P_{o_2}}{P_{o_1}}=\left[\frac{\dfrac{2\gamma M_1^2\cancel{-(\gamma-1)}+\frac{1}{2}(\gamma-1)^2M_1^2\cancel{+(\gamma-1)}}{2\gamma M_1^2-(\gamma-1)}}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}}\frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)}$$

$$=\left[\frac{M_1^2\overbrace{\left(2\gamma+\frac{1}{2}(\gamma-1)^2\right)}^{\frac{(\gamma+1)(\gamma+1)}{2}}}{\left(1+\frac{\gamma-1}{2}M_1^2\right)\left(2\gamma M_1^2-(\gamma-1)\right)}\right]^{\frac{\gamma}{\gamma-1}}\frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)}$$

$$=\left[\frac{\frac{M_1^2}{2}(\gamma+1)}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{\gamma+1}{2\gamma M_1^2-(\gamma-1)}\right]^{-\frac{\gamma}{\gamma-1}}\frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)}$$

$$\boxed{\frac{P_{o_2}}{P_{o_1}}=\left[\frac{\frac{\gamma+1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}}\Bigg/\left[\frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)}\right]^{\frac{1}{\gamma-1}}}\quad\text{------}\;(4.9)$$

The entropy change across the shock wave can be obtained from equation (1.7)

$$S_2-S_1=C_p\ln\left(\frac{T_2}{T_1}\right)-R\ln\left(\frac{P_2}{P_1}\right)\qquad C_p=C_v+R$$

$$S_2 - S_1 = (C_v + R)\ln\left(\frac{P_2}{P_1}\frac{\rho_1}{\rho_2}\right) - R\ln\left(\frac{P_2}{P_1}\right) \qquad \div R$$

$$\frac{S_2-S_1}{R} = \left(1+\frac{1}{\gamma-1}\right)\ln\left(\frac{P_2}{P_1}\frac{\rho_1}{\rho_2}\right) - \ln\left(\frac{P_2}{P_1}\right)$$

$$= \ln\left(\frac{P_2}{P_1}\frac{\rho_1}{\rho_2}\right)^{+\frac{\gamma}{\gamma-1}} - \ln\left(\frac{P_2}{P_1}\right)$$

$$= \ln\left[\left(\frac{P_2}{P_1}\frac{\rho_1}{\rho_2}\right)^{+\frac{\gamma}{\gamma-1}}\left(\frac{P_2}{P_1}\right)^{-1}\right]$$

$$\boxed{\frac{S_2-S_1}{R} = \ln\left[\left(\frac{P_2}{P_1}\right)^{\frac{1}{\gamma-1}}\left(\frac{\rho_2}{\rho_1}\right)^{-\frac{\gamma}{\gamma-1}}\right]} \quad \text{--------- (4.10)}$$

substitution of equs. (4.7) and (4.8) into equ. (4.10) gives :-

$$\boxed{\frac{S_2-S_1}{R} = \ln\left[\left(\frac{2\gamma M_1^2-(\gamma-1)}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\left(\frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}\right)^{-\frac{\gamma}{\gamma-1}}\right]} \quad \text{---- (4.11)}$$

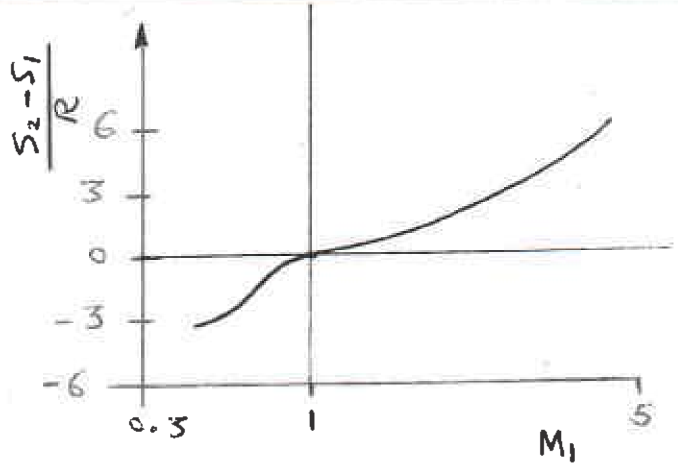


Figure (4.4) Variation of entropy change across normal shock with upstream Mach number (M_1) for ($\gamma = 1.4$).

The entropy must increased. It with be seen that this can only if

$$M_1 > 1$$

∴ The flow ahead the shock wave must be always <u>supersonic</u> and the shock must be compressive ($P_2/P_1 > 1$).

For isentropic flow $\longrightarrow$ (A^*) is used as a reference area.
For normal shock flow areas downstream of the shock cannot be referenced to the critical area upstream of the shock.
So that $A_1^* \neq A_2^*$.

For steady flow acerss the shock wave

$$\dot{m}_1 = \dot{m}_2 = \text{Constant}$$

with the aid of equ. (3.16)

$$A_1^* P_{o_1} \cancel{\sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}} \frac{\gamma}{R T_{o_1}}}} = A_2^* P_{o_2} \cancel{\sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}} \frac{\gamma}{R T_{o_2}}}}$$

$T_{o_1} = T_{o_2}$ (adiabatic flow)

$$\therefore \boxed{\frac{A_2^*}{A_1^*} = \frac{P_{o_1}}{P_{o_2}}} \quad \text{------} \quad (4.12)$$

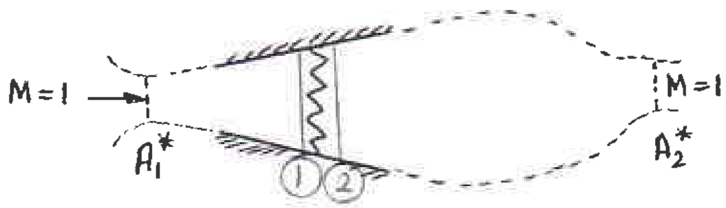


Figure (4.5)

4.3. Non-Isentropic Flow in Converging-Diverging Nozzle

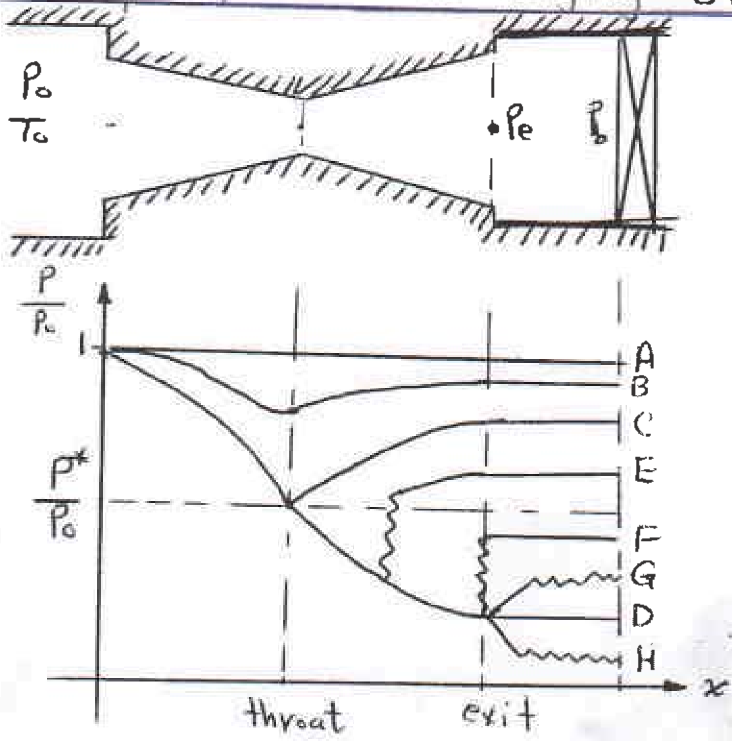


Figure (4.6)