Chapter - 4 -
Shock Waves

4.1. Introduction

It has been found experimentally, it is possible for a spontaneous change to occur in a flow, the velocity decreasing and the pressure increasing, this change is termed a shock wave.

Shock wave can only occur if the initial flow is supersonic.

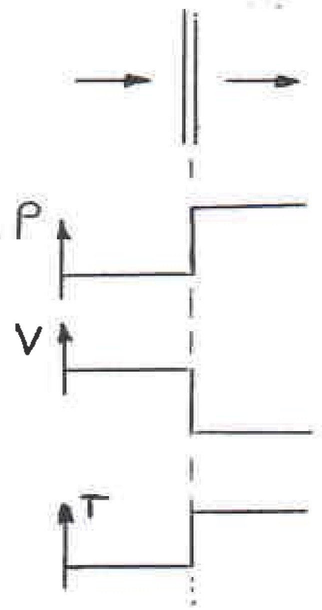


Figure (4.1) Changes through a Normal Shock Wave.

Shock Waves → Normal
Shock Waves → oblique
Shock Waves → Curved

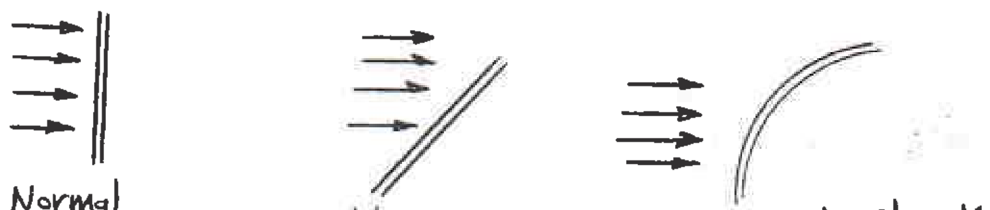


Figure (4.2) Shock Waves

4.2 Governing Equations of the Stationary Normal Shock Waves

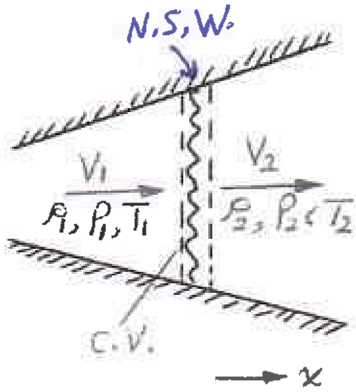


Figure (4-3)

Assumptions

1- The normal shock wave (N.S.W.) takes place at constant cross-sectional area. (Shock thickness is very small $\simeq 0.25-1\ \mu m$) $\Rightarrow A_1 = A_2$

2- Flow is adiabatic

Continuity Equation $\dot{m} = \rho V A$

$$\rho_1 V_1 \cancel{A_1} = \rho_2 V_2 \cancel{A_2}$$

$$\rho_1 V_1 = \rho_2 V_2 \text{ ---------- } (4.1)$$

Momentum Equation (x-momentum) $\sum F = \dot{m} V_2 - \dot{m} V_1$

$$P_1 \cancel{A_1} - P_2 \cancel{A_2} = \rho_2 \cancel{A_2} V_2^2 - \rho_1 \cancel{A_1} V_1^2$$

$$P_2 + \rho_2 V_2^2 = P_1 + \rho_1 V_1^2$$

Noting that the term ρV^2 in the momentum equation

$$\rho V^2 = \frac{P}{\cancel{RT}} M^2 \gamma \cancel{RT} = P \gamma M^2$$

$$\therefore P_2 + P_2 \gamma M_2^2 = P_1 + P_1 \gamma M_1^2 \xrightarrow{\text{or}} P_2 (1+\gamma M_2^2) = P_1 (1+\gamma M_1^2)$$

$$\boxed{\frac{P_2}{P_1} = \frac{(1+\gamma M_1^2)}{(1+\gamma M_2^2)}} \text{ ------- } (4.2)$$

Energy Equation

$$\cancel{q} = \left(h_2 + \frac{V_2^2}{2}\right) - \left(h_1 + \frac{V_1^2}{2}\right) \quad \text{For ideal gas } h = C_p T$$

$$\therefore C_p T_2 + \frac{V_2^2}{2} = C_p T_1 + \frac{V_1^2}{2} \quad \text{But } C_p = \frac{\gamma R}{\gamma - 1} \ \& \ M = \frac{V}{\sqrt{\gamma R T}}$$

$$\boxed{\frac{T_2}{T_1} = \frac{1 + \frac{\gamma - 1}{2} M_1^2}{1 + \frac{\gamma - 1}{2} M_2^2}} \text{ ------ } (4.3)$$

Density relation follows from the ideal gas equation

$$\frac{\rho_2}{\rho_1} = \frac{P_2}{\cancel{R} T_2} \frac{\cancel{R} T_1}{P_1} = \frac{P_2}{P_1} \frac{T_1}{T_2} \text{ ------ } (4.4)$$

$\because \rho V = \frac{P}{RT} M\sqrt{\gamma R T}$, equation (4.1) can be written as:-

$$\frac{P_1}{\sqrt{T_1}} M_1 = \frac{P_2}{\sqrt{T_2}} M_2 \longrightarrow \sqrt{\frac{T_2}{T_1}} = \frac{M_2}{M_1}\,\frac{P_2}{P_1}$$

Substitution of equs. (4.2) and (4.3) in the above equation gives:-

$$\left[\frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2}\right]^{1/2} = \left(\frac{M_2}{M_1}\right)\left(\frac{(1+\gamma M_1^2)}{(1+\gamma M_2^2)}\right)$$

Squaring, we obtain

$$\frac{2+(\gamma-1)M_1^2}{2+(\gamma-1)M_2^2} = \left(\frac{M_2}{M_1}\right)^2\left[\frac{1+2\gamma M_1^2+\gamma^2 M_1^4}{\underbrace{1+2\gamma M_2^2+\gamma^2 M_2^4}}\right]$$

or

$$(2M_2^2+(\gamma-1)M_2^4)+(2M_2^2+(\gamma-1)M_2^4)(2\gamma M_1^2+\gamma^2 M_1^4) =$$
$$(2M_1^2+(\gamma-1)M_1^4)+(2M_1^2+(\gamma-1)M_1^4)(2\gamma M_2^2+\gamma^2 M_2^4)$$

or

$$(\gamma-1)(M_2^4-M_1^4)+2(M_2^2-M_1^2)+M_1^2M_2^2\left[((2+(\gamma-1)M_2^2)(2\gamma+\gamma^2M_1^2))-\right.$$
$$\left.((2+(\gamma-1)M_1^2)(2\gamma+\gamma^2M_2^2))\right]=0$$

or

$$(\gamma-1)(M_2^4-M_1^4)+2(M_2^2-M_1^2)+M_2^2M_1^2\left[(\cancel{4\gamma}+2\gamma^2M_1^2+2\gamma(\gamma-1)M_2^2+\cancel{\gamma^2(\gamma-1)M_1^2M_2^2})\right.$$
$$\left.-(\cancel{4\gamma}+2\gamma^2M_2^2+2\gamma(\gamma-1)M_1^2+\cancel{\gamma^2(\gamma-1)M_1^2M_2^2})\right]=0$$

or

$$(\gamma-1)(M_2^4-M_1^4)+2(M_2^2-M_1^2)+M_2^2M_1^2\left[(\cancel{2\gamma^2M_1^2}+\cancel{2\gamma^2M_2^2}-2\gamma M_2^2)\right.$$
$$\left.-(\cancel{2\gamma^2M_2^2}+\cancel{2\gamma^2M_1^2}-2\gamma M_1^2)\right]=0$$

$$\therefore (\gamma-1)(M_2^4-M_1^4)-2\gamma M_1^2M_2^2(M_2^2-M_1^2)+2(M_2^2-M_1^2)=0$$

or

$$(M_2^2-M_1^2)\left((\gamma-1)(M_2^2+M_1^2)-2\gamma M_1^2M_2^2+2\right)=0$$

Sol 1

$M_2^2 = M_1^2$ this means that there is no change in the Mach number across the shock wave, so that this solution is trivial.

Sol 2

$$(\gamma-1)(M_2^2+M_1^2) - 2\gamma M_1^2 M_2^2 + 2 = 0$$

this gives :-

$$\boxed{M_2^2 = \frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)}} \quad \text{-------} (4.5)$$

substitution of equ.(4.5) into equ.(4.3) gives :-

$$\frac{T_2}{T_1} = \frac{2+(\gamma-1)M_1^2}{\left(2+(\gamma-1)\left(\frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)}\right)\right)}$$

$$\boxed{\frac{T_2}{T_1} = \frac{(2+(\gamma-1)M_1^2)(2\gamma M_1^2-(\gamma-1))}{(\gamma+1)^2 M_1^2}} \quad \text{-------} (4.6)$$

And substitution of equ.(4.5) into equ.(4.2) gives :-

$$\frac{P_2}{P_1} = \frac{1+\gamma M_1^2}{\left[1+\gamma\left(\frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)}\right)\right]} = \frac{(1+\gamma M_1^2)(2\gamma M_1^2-(\gamma-1))}{(2\gamma M_1^2-(\gamma-1))+\gamma(\gamma-1)M_1^2+2\gamma}$$

$$= \frac{(1+\gamma M_1^2)(2\gamma M_1^2-(\gamma-1))}{2\gamma M_1^2-(\gamma-1)+\gamma^2 M_1^2-\gamma M_1^2+2\gamma} = \frac{(1+\gamma M_1^2)(2\gamma M_1^2-(\gamma-1))}{\underbrace{\gamma^2 M_1^2+\gamma M_1^2+\gamma+1}_{(\gamma+1)M_1^2\gamma}}$$

$$\frac{P_2}{P_1} = \frac{\cancel{(\gamma M_1^2+1)}(2\gamma M_1^2-(\gamma-1))}{(\gamma+1)\cancel{(\gamma M_1^2+1)}} = \boxed{\frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)}} \quad \text{------} (4.7)$$

substitution of equs. (4.6) and (4.7) into equ.(4-4) gives :-

$$\frac{\rho_2}{\rho_1} = \frac{\cancel{2\gamma M_1^2-(\gamma-1)}}{(\gamma+1)} \cdot \frac{(\gamma+1)^2 M_1^2}{(2+(\gamma-1)M_1^2)\cancel{(2\gamma M_1^2-(\gamma-1))}}$$

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}} \quad \text{------} (4.8)$$

The stagnation pressure ratio across the normal shock wave is obtained by:-

$$\frac{P_{o_2}}{P_{o_1}} = \frac{P_{o_2}}{P_2}\,\frac{P_1}{P_{o_1}}\,\frac{P_2}{P_1} = \frac{P_{o_2}/P_2}{P_{o_1}/P_1}\,\frac{P_2}{P_1}$$

equ.(4.7)

$$\frac{P_{o_2}}{P_{o_1}} = \left[\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}} \frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)}$$

sub. (M_2^2) from equ.(4.5) in the above equation, we get :-

M_2^2

$$\frac{P_{o_2}}{P_{o_1}} = \left[\frac{1+\left(\frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)}\right)\frac{\gamma-1}{2}}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}} \frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)}$$

$$\frac{P_{o_2}}{P_{o_1}} = \left[\frac{\dfrac{2\gamma M_1^2-\cancel{(\gamma-1)}+\frac{1}{2}(\gamma-1)^2M_1^2+\cancel{(\gamma-1)}}{2\gamma M_1^2-(\gamma-1)}}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}} \frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)}$$

$\frac{(\gamma+1)(\gamma+1)}{2}$

$$= \left[\frac{M_1^2\left(2\gamma+\frac{1}{2}(\gamma-1)^2\right)}{\left(1+\frac{\gamma-1}{2}M_1^2\right)\left(2\gamma M_1^2-(\gamma-1)\right)}\right]^{\frac{\gamma}{\gamma-1}} \frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)}$$

$$= \left[\frac{\frac{M_1^2}{2}(\gamma+1)}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}} \left[\frac{\gamma+1}{2\gamma M_1^2-(\gamma-1)}\right]^{-\frac{\gamma}{\gamma-1}} \frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)}$$

$$\boxed{\frac{P_{o_2}}{P_{o_1}} = \left[\frac{\frac{\gamma+1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}} \Bigg/ \left[\frac{2\gamma M_1^2-(\gamma-1)}{(\gamma+1)}\right]^{\frac{1}{\gamma-1}}} \quad ------ (4.9)$$

The entropy change across the shock wave can be obtained from equation (1.7)

$$S_2 - S_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right) \qquad c_p = c_v + R$$

$$S_2 - S_1 = (C_v + R)\ln\left(\frac{P_2}{P_1}\frac{\rho_1}{\rho_2}\right) - R\ln\left(\frac{P_2}{P_1}\right) \qquad \div R$$

$$\frac{S_2-S_1}{R} = \left(1+\frac{1}{\gamma-1}\right)\ln\left(\frac{P_2}{P_1}\frac{\rho_1}{\rho_2}\right) - \ln\left(\frac{P_2}{P_1}\right)$$

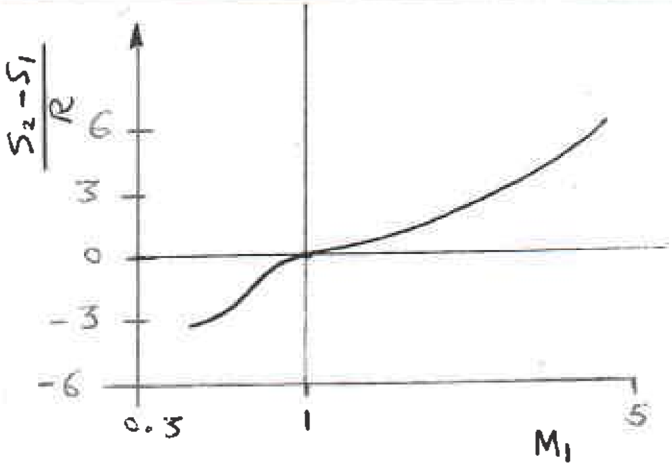
$$= \ln\left(\frac{P_2}{P_1}\frac{\rho_1}{\rho_2}\right)^{+\frac{\gamma}{\gamma-1}} - \ln\left(\frac{P_2}{P_1}\right)$$

$$= \ln\left[\left(\frac{P_2}{P_1}\frac{\rho_1}{\rho_2}\right)^{+\frac{\gamma}{\gamma-1}}\left(\frac{P_2}{P_1}\right)^{-1}\right]$$

$$\boxed{\frac{S_2-S_1}{R} = \ln\left[\left(\frac{P_2}{P_1}\right)^{\frac{1}{\gamma-1}}\left(\frac{\rho_2}{\rho_1}\right)^{-\frac{\gamma}{\gamma-1}}\right]} \quad \text{--------- (4.10)}$$

substitution of equs. (4.7) and (4.8) into equ. (4.10) gives :-

$$\boxed{\frac{S_2-S_1}{R} = \ln\left[\left(\frac{2\gamma M_1^2-(\gamma-1)}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\left(\frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}\right)^{-\frac{\gamma}{\gamma-1}}\right]} \quad \text{---- (4.11)}$$



Figure(4.4) Variation of entropy change across normal shock with upstream Mach number (M_1) for ($\gamma=1.4$).

The entropy must increased. It with be seen that this can only if

$$M_1 > 1$$

∴ The flow ahead the shock wave must be always <u>supersonic</u> and the shock must be compressive ($P_2/P_1 > 1$).

For isentropic flow $\longrightarrow$ (A^*) is used as a reference area.
For normal shock flow areas downstream of the shock cannot be referenced to the critical area upstream of the shock.
So that $A_1^* \neq A_2^*$.

For steady flow acerross the shock wave

$$\dot{m}_1 = \dot{m}_2 = \text{Constant}$$

With the aid of equ. (3.16)

$$A_1^* P_{o_1} \sqrt{\cancel{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} \frac{\gamma}{R T_{o_1}}} = A_2^* P_{o_2} \sqrt{\cancel{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}} \frac{\gamma}{R T_{o_2}}} \qquad T_{o_1} = T_{o_2} \text{ (adiabatic flow)}$$

$$\therefore \boxed{\frac{A_2^*}{A_1^*} = \frac{P_{o_1}}{P_{o_2}}} \quad \text{------} \quad (4.12)$$

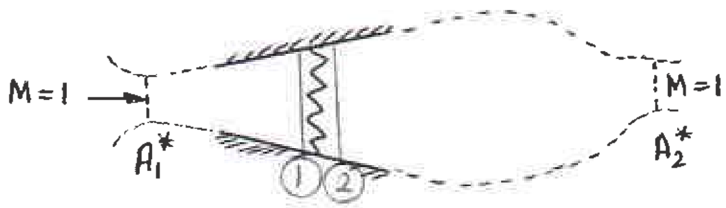


Figure (4.5)

4.3. Non-Isentropic Flow in Converging-Diverging Nozzle

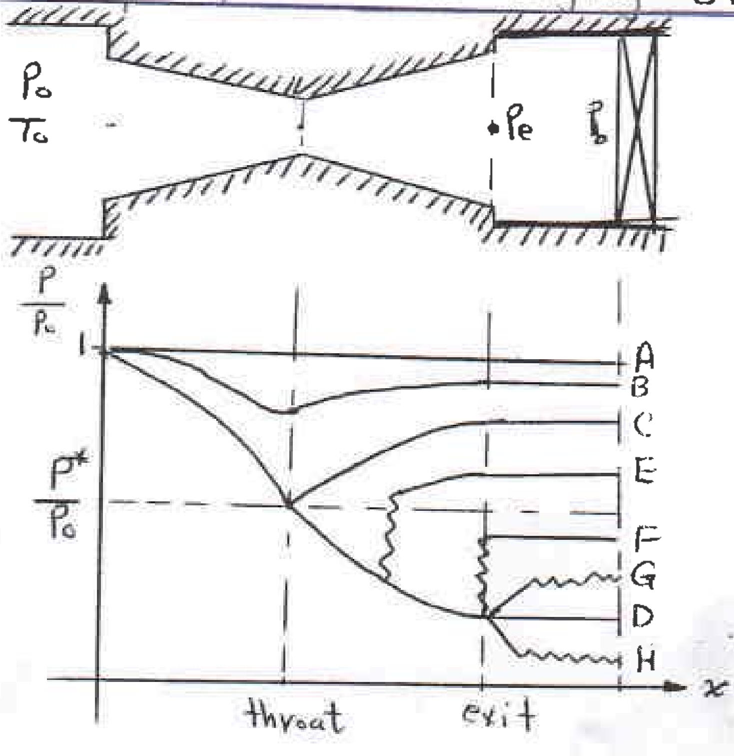
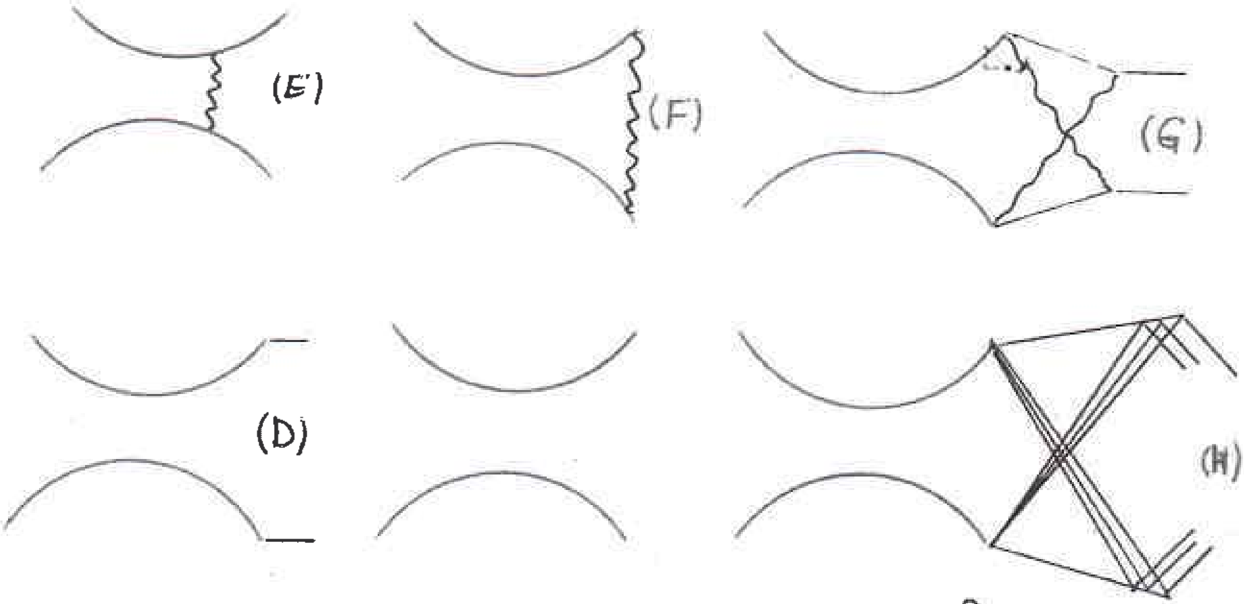


Figure (4.6)

Curve E Normal shock in the divergent part
Curve F Normal shock at the exit plane.
Curve D Perfectly expandend (design)
Curve G overexpanded (oblique shock waves)
Curve H underexpanded (expansion waves).



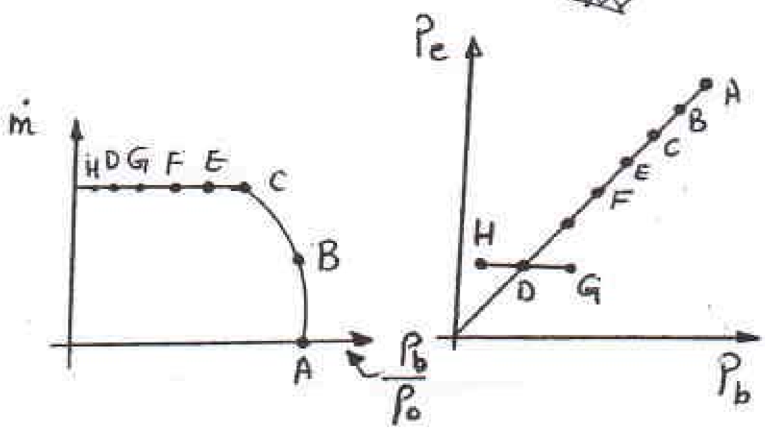


Figure (4.7)

4.4 Normal Shock Wave Table ($\gamma = 1.4$)

M_1	M_2	P_2/P_1	T_2/T_1	ρ_2/ρ_1	P_{o_2}/P_{o_1}

4.5. Moving Normal Shock Wave

In most cases the shock wave moves such as the shock that stands on a moving body or generated from an explosion or the one in a shock tube. Take a normal shock wave that moves with constant velocity (V_s) towords a stationary gas a shown in Figure (4.9a). If a control volume is fixed on the shock as shown in Figure(4.9.b), the relative motion of the gas will be considered.

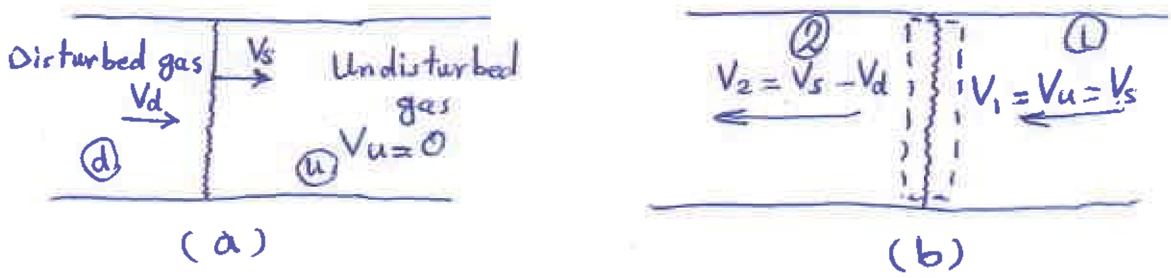


Figure 4.9: Moving shock wave. (a) original case (b) Transformed case.

4.6. Shock Wave Strength(β)

It's a measure of the pressure increase across the shock.

$$\beta = \frac{P_2}{P_1} - 1 \qquad (4.13)$$

Substituting Eq. (4.7) into Eq. (4.13)

$$\beta = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1} - 1$$

$$\therefore \boxed{\beta = \frac{2\gamma}{\gamma+1}(M_1^2 - 1)} \qquad (4.14)$$

4.7. Oblique Shock Wave (O.S.W.)

Oblique Shock wave appears when the surface changes its direction such as at airplane nose for supersonic flow. However, a supersonic airplane does not necessarily generate an oblique shock that is attached to its nose. Instead, it may have a detached N.S.W. ahead of the airplane. As the airplane accelerates to its supersonic cruising speed the flow will develop from subsonic, through supersonic with a detached N.S.W. to attached Oblique Shock waves as shown in Figure

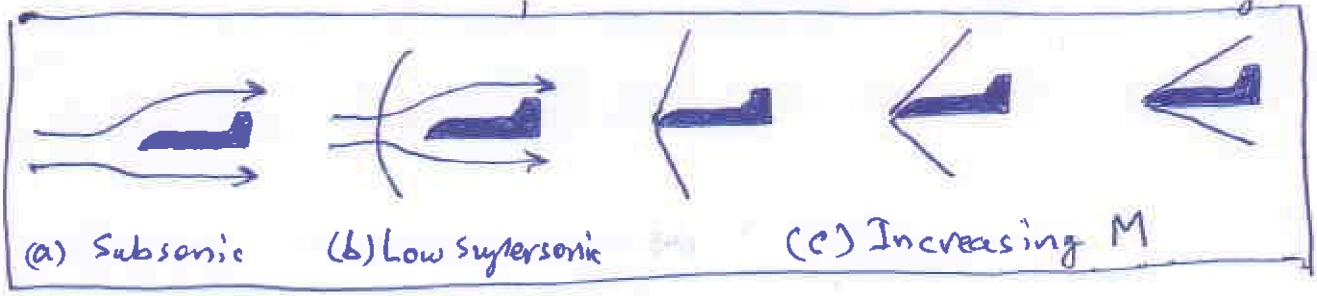


Figure 4.10. Development of shock wave for airplane with increasing speed.

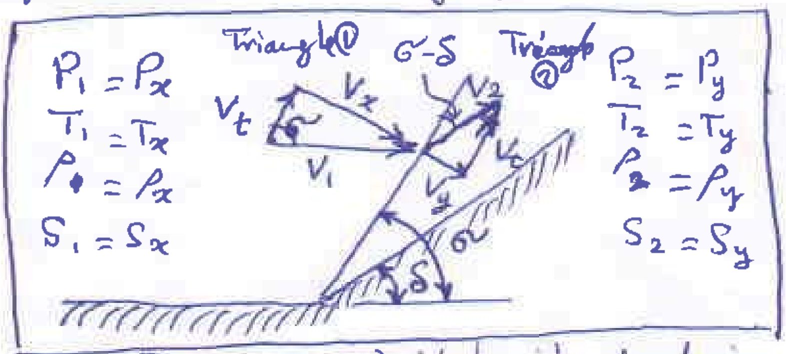


Figure (4.11) Velocity Analysis in Oblique Shock Wave

From triangle ①

$$V_x = V_1 \sin \sigma$$

$$\therefore M_x = M_1 \sin \sigma \quad \text{—— (4.15)}$$

From triangle ②

$$V_y = V_2 \sin(\sigma - \delta)$$

$$\therefore M_y = M_2 \sin(\sigma - \delta) \quad \text{—— (4.16)}$$

$$\Delta S = \cancel{C_p \ln \frac{T_{o2}}{T_{o1}}}^{\,0} - R \ln \frac{P_{o2}}{P_{o1}} \quad \text{(adiabatic)} \Rightarrow S_2 - S_1 = -R \ln \frac{P_{o2}}{P_{o1}}$$

$$\Delta S = \cancel{C_p \ln \frac{T_{oy}}{T_{ox}}}^{\,0} - R \ln \frac{P_{oy}}{P_{ox}} \quad \text{(adiabatic)} \Rightarrow S_y - S_x = -R \ln \frac{P_{oy}}{P_{ox}}$$

but $S_y - S_x = S_2 - S_1$

$$\therefore \frac{P_{oy}}{P_{ox}} = \frac{P_{o2}}{P_{o1}} \quad \text{———— (4.17)}$$

But $P_{oy} \neq P_{o2}$ & $P_{ox} \neq P_{o1}$

$$\text{at } M_x \xrightarrow{\text{N.S.W. table}} \frac{P_y}{P_x} = \frac{P_2}{P_1}, \quad \frac{T_y}{T_x} = \frac{T_2}{T_1}, \quad \frac{\rho_y}{\rho_x} = \frac{\rho_2}{\rho_1} \quad \text{---- etc.} \quad \text{—— (4.18)}$$

To find the relation between the angles σ & δ, the velocity component triangles upstream and downstream can be used as follows:

$$\tan \sigma = \frac{V_x}{V_t}$$

$$\tan(\sigma - \delta) = \frac{V_y}{V_t}$$

By dividing the second above equation to the first equation above, the following equation is obtained

$$\tan \delta = 2 \cot \sigma \frac{M_1^2 \sin^2 \sigma - 1}{M_1^2 (\gamma + \cos 2\sigma) + 2} \quad \text{——— (4.19)}$$

This relation is represented by the following figure (4.12)

$\therefore M_1 \geq 1$ for N.S.W.

$\therefore M_1 \sin \sigma \geq 1$ for O.S.W.

$$\sin \sigma \geq \frac{1}{M_1}$$

$$\sigma \geq \sin^{-1} \frac{1}{M_1} \quad \text{——— (4.20)}$$

For a certain value of M_1, the minimum value of σ is ($\sigma_{min} = \sin^{-1} \frac{1}{M_1}$) the Mach angle and the shock wave is Mach wave. On the other hand, the maximum value of σ is 90° (the wave is N.S.W)

$$\sin^{-1} \frac{1}{M_1} \leq \sigma \leq 90^\circ \quad \text{——— (4.21)}$$

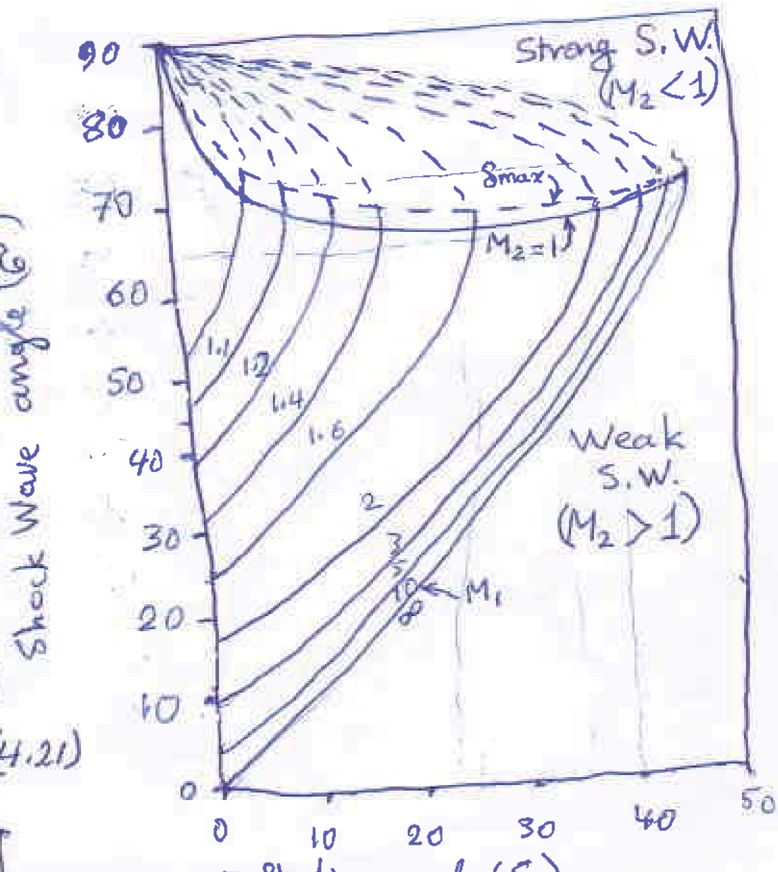


Figure (4.12) Oblique shock Chart

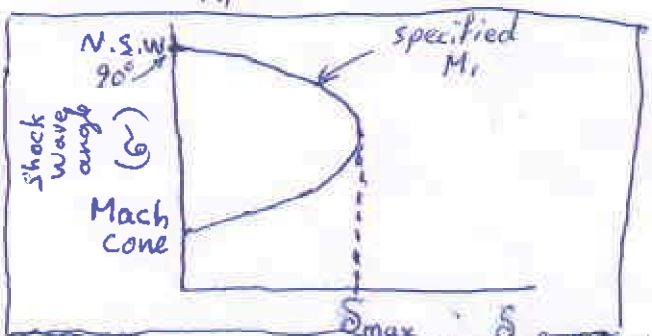


Figure (4.13) Limits of O.S.W. angle

There are two cases :

① If $\delta > \delta_{max}$, there is no solution as shown in Figure (4.12), while a detached bow O.S.W. is appeared a head as shown in Figure (4.14).

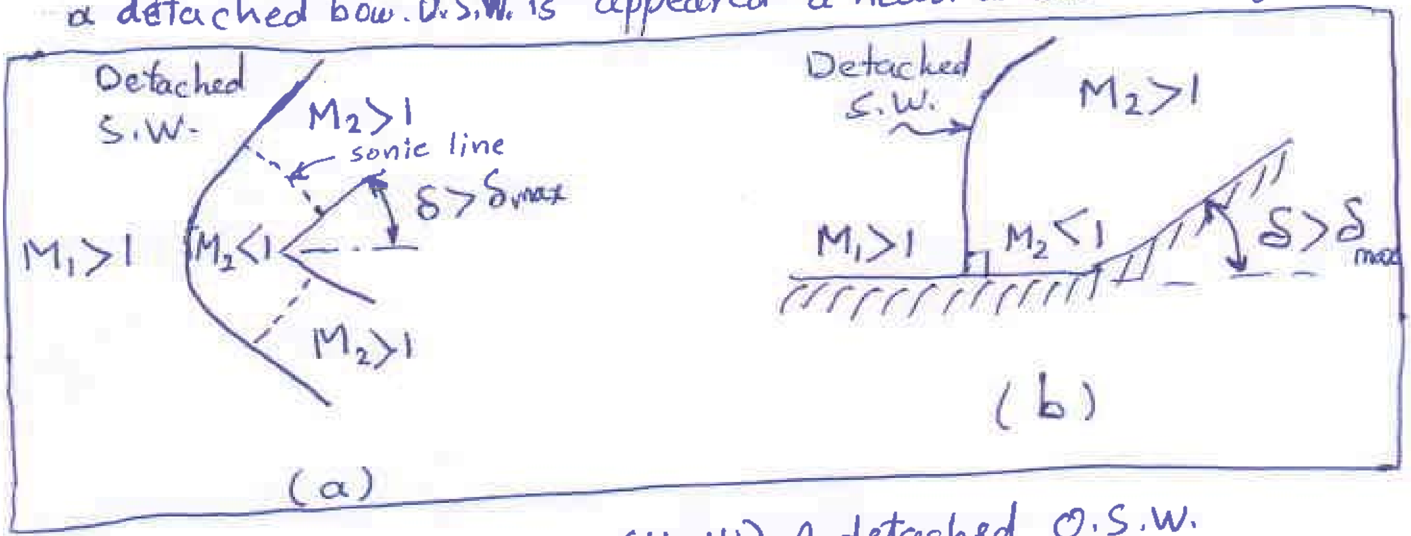


Figure (4.14) A detached O.S.W. ahead (a) a wedge & (b) 2D Passage with concave sudden turn

Same thing happens when $M_1 < M_{1\,min}$ where $M_{1\,min}$ is the value of M_1 at which O.S.W. is attached the leading edge of the wedge or the deflection in a 2-D passage. The region down stream the detached O.S.W. is very complicated where both subsonic & supersonic field are existed. The detached bow O.S.W. is strong ($M_2<1$) as σ is large while it is weak ($M_2>1$) as σ has a small value.

② If $\delta < \delta_{max}$ ($M_1 > M_{1\,min}$) there are two solutions : Strong shock solution when σ has a high value & weak shock solution when σ is at its low value as descriped in Figure (4.15).

Experimentally, it is found that for a given M_1 & δ in external flows the shock wave is weak.

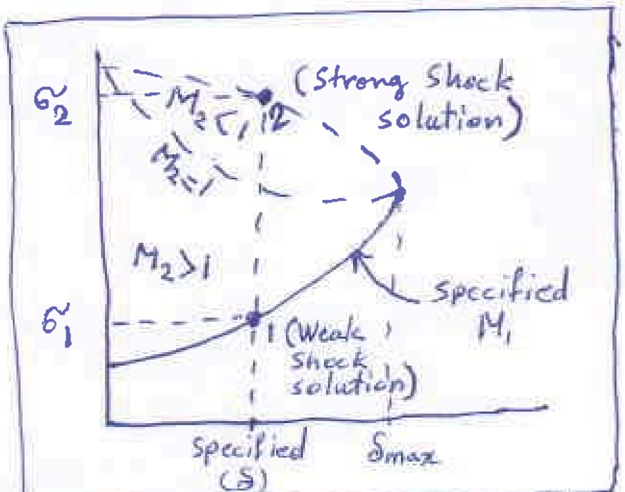


Figure 4.15 Strong & Weak O.S.W. solutions

For external flow such the flow over a wedge shown in Figure (4.16), either both solutions are considered, weak wave solution is considered if the experimental experience is adopted (mentioned in the problem), strong wave solution is considered if it is required in the problem or the solution can be concluded:

① if $M_2 < 1 \Rightarrow$ strong wave solution

② if there are series of waves (shock system) $\Rightarrow$ weak wave solution.

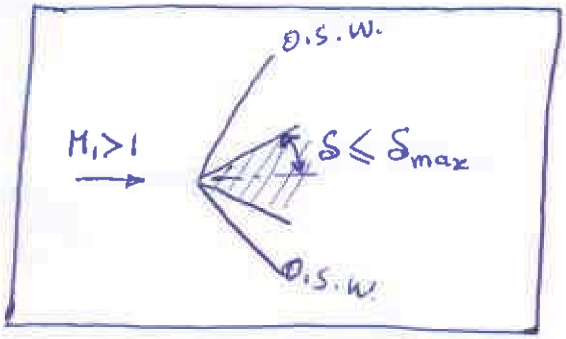


Figure 4.16. Attached O.S.W. over a wedge

For a supersonic flow through a 2-D passage with sudden turn with $\delta \leq \delta_{max}$ (or $M_1 \geq M_{1,min}$), the two solutions are described as follows:

① Strong wave solution

If the downstream pressure is sufficiently high, an O.S.W. will appear attached to the lower wall and N.S.W. to the upper wall ($\delta = 0$) and the flow is non uniform downstream the shock wave as shown in Figure (4.17)

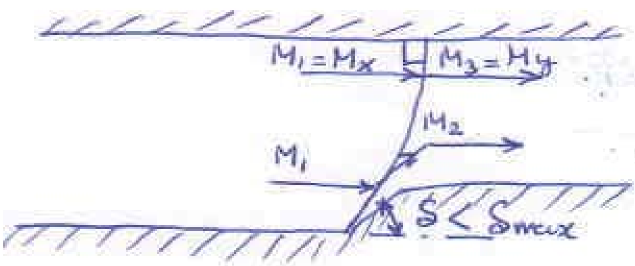
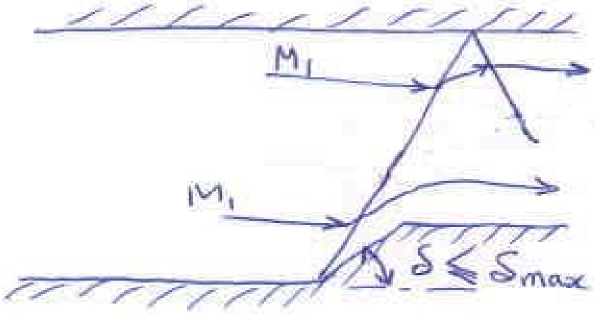


Figure (4.17) Strong O.S.W. in a 2-D passage with sudden turn.

② Weak Wave Solution

The wave stay's straight and it is reflected from the upper wall at different σ. The flow down stream the weak O.S.W. remains uniform as shown in Figure 4.18



supersonic