CH. 5 : Constant Area Duct with Friction or with Heat Transfer

5.1. Governing Equations of Flow in Constant Area Duct with Friction

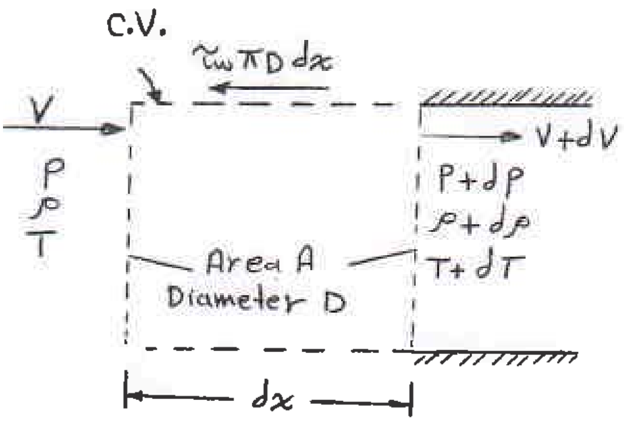


Figure (5.1) C.V. for flow in a constant area duct with friction.

Continuity

$$\rho V A = \dot m \longrightarrow \rho V = \frac{\dot m}{A} = G = \text{constant} \longrightarrow \frac{d\rho}{\rho} + \frac{dV}{V} = 0 \quad \text{------ (5.1)}$$

x - Momentum

$$PA - (P+dP)\overbrace{A}^{\frac{\pi}{4}D^2} - \tau_w \pi D\, dx = \dot m(\cancel{V} + dV - \cancel{V})$$

$$\text{or}\quad dP + \frac{4\tau_w\, dx}{D} + \rho V\, dV = 0 \quad \text{----------- (5.2)}$$

Energy Equation

$$h + \frac{1}{2}V^2 = h_o = C_p T_o = C_p T + \frac{V^2}{2} = \text{constant}$$

$$\text{or}\quad C_p\, dT + V\, dV = 0 \quad \text{------ (5.3)}$$

Perfect Gas Law

$$P = \rho R T \quad \text{or} \quad \frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T} \quad \text{----- (5.4)}$$

Fanning Friction Factor

$$\tau_w = \frac{1}{2} f \rho V^2 = \frac{1}{2} f \gamma P M^2 \quad \text{-------- (5.5)}$$

$$\therefore V^2 = \gamma R T M^2 \quad \text{or} \longrightarrow \underbrace{\frac{dV^2}{V^2}}_{\frac{2dV}{V}} = \frac{dM^2}{M^2} + \frac{dT}{T} \quad \text{------ (5.6)}$$

Equation (5.2) can be written as

$$dP = -\frac{2f\gamma P M^2 dx}{D} - \rho V dV \qquad \div P$$

$$\frac{dP}{P} = -2\gamma M^2 \frac{f}{D} dx - \overset{\frac{1}{RT}}{\frac{\rho}{P}} V dV = -2\gamma M^2 \frac{f}{D} dx - \frac{\gamma V^2}{\gamma R T} \frac{dV}{V}$$

$$\frac{dP}{P} = -2\frac{f}{D}\gamma M^2 dx - \gamma M^2 \frac{dV}{V} \qquad \text{------} (5.7)$$

rewrite equation (5.4) as

$$\frac{dP}{P} = \overset{d\rho/\rho}{-\frac{dV}{V}} + \frac{dT}{T}$$

From equation (5.3), we can write $\quad dT + \frac{V}{\varphi} dV = 0 \quad \div T$

$$\frac{dT}{T} = -\frac{V}{\varphi T} dV = -\frac{V^2 \gamma R}{\varphi T \gamma R}\frac{dV}{V} = -(\gamma - 1) M^2 \frac{dV}{V}$$

$$\therefore \frac{dP}{P} = -\frac{dV}{V} - (\gamma-1)M^2 \frac{dV}{V} \qquad \text{------} (5.8)$$

~~By equating~~ Equate eqs. (5.7) and (5.8), we get

$$-2\frac{f}{D}\gamma M^2 dx \cancel{- \gamma M^2 \frac{dV}{V}} = -\frac{dV}{V} \cancel{- \gamma M^2 \frac{dV}{V}} + M^2 \frac{dV}{V}$$

$$\therefore \boxed{\frac{dV}{V} = 2f \frac{dx}{D} \frac{\gamma M^2}{(1 - M^2)}} \qquad \text{------} (5.9)$$

Substitution of equation (5.9) into (5.8), gives

$$\boxed{\frac{dP}{P} = -\frac{[1 + (\gamma-1)M^2]\gamma M^2}{(1-M^2)} 2f \frac{dx}{D}} \qquad \text{------} (5.10)$$

and

$$\boxed{\frac{dT}{T} = -(\gamma-1)M^2 \frac{dV}{V} = -\frac{(\gamma-1)\gamma M^4}{(1-M^2)} 2f \frac{dx}{D}} \qquad \text{------} (5.11)$$

Substitution of equ. (5.9) into equ. (5.1), gives:-

$$\boxed{\frac{d\rho}{\rho} = -2f \frac{dx}{D} \frac{\gamma M^2}{(1-M^2)}} \qquad \text{------} (5.12)$$

Substitution of eqs. (5.9) and (5.11) into equ. (5.6), gives:-

$$\boxed{\frac{dM^2}{M^2} = \frac{\gamma M^2\left[1+\frac{\gamma-1}{2}M^2\right]}{(1-M^2)}\,4f\,\frac{dx}{D}} \quad \text{------- (5.13)}$$

property	Subsonic	Supersonic
V ⟶	increases ⟶	decreases
P ⟶	decreases ⟶	increases
T ⟶	decreases ⟶	increases
ρ ⟶	decreases ⟶	increases
M ⟶	increases ⟶	decreases

5.2. Fanno Line

From 1st & 2nd Laws of Thermodynamics, the following Eq. can be derived,

$$ds = c_p\frac{dT}{T} - \frac{R}{P}dP \qquad \div c_p \qquad \frac{R}{c_p} = \frac{\gamma-1}{\gamma}$$

$$\frac{ds}{c_p} = \frac{dT}{T} - \frac{\gamma-1}{\gamma}\,\frac{dP}{P} \quad \text{------- (5.14)}$$

from energy equation $c_p dT + V dV = 0 \longrightarrow \frac{dV}{V} = -\frac{c_p dT}{V^2}$ ---- (5.15)

From equation (5.4) and (5.1) and (5.15)

$$\frac{dP}{P} = \frac{dT}{T} + \frac{c_p}{V^2}dT \quad \text{----- (5.16)}$$

substitution of equ. (5.16) into (5.14) gives

$$\frac{ds}{c_p} = \frac{dT}{T} - \frac{\gamma-1}{\gamma}\left(\frac{dT}{T} + \frac{c_p}{V^2}dT\right) \quad \text{------ (5.17)}$$

$$\because\ T_o = T + \frac{V^2}{2c_p} \longrightarrow V^2 = 2c_p(T_o - T) \quad \text{---- (5.18)}$$

From equ. (5.17) and (5.18) we have:-

$$\frac{ds}{c_p} = \left(1 - \frac{\gamma-1}{\gamma}\right)\frac{dT}{T} - \frac{\gamma-1}{\gamma}\,\frac{\cancel{c_p}}{2\cancel{c_p}(T_o-T)}\,dT$$

or

$$\frac{ds}{c_p} = \frac{1}{\gamma}\,\frac{dT}{T} - \frac{\gamma-1}{2\gamma}\,\frac{dT}{T_o-T} \quad \text{------ (5.19)}$$

By Integrating equation (5.19) between arbitrary state (T_1, S_1) and any Value of (T) and (S), we have :-

$$\frac{S-S_1}{C_p} = \ln\left[\left(\frac{T}{T_1}\right)^{\frac{1}{\gamma}}\left(\frac{T_o-T}{T_o-T_1}\right)^{\frac{\gamma-1}{2\gamma}}\right] \quad \text{------- (5.20)}$$

Equation (5.20) is the equation of Fanno line for a perfect gas.

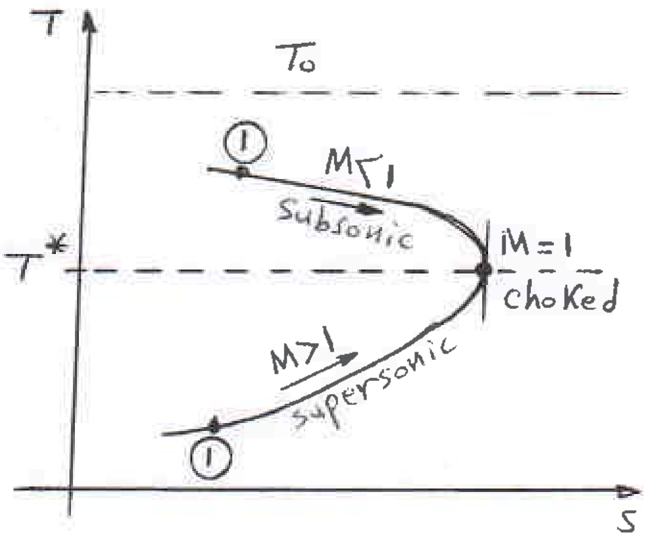


Figure (5.2) Fanno Line

According to 2nd law, entropy must be increase, so that the path of states along any one of the Fanno lines must be towards right.

Case 1 subsonic The effects of friction tend to increase Mach number and to decrease the temperature.
So that the upper part of Fanno curve is for subsonic flow.

Case 2 Supersonic The effects of friction tend to decrease Mach number and to increase the temperature.
So that the lower part of Fanno curve is for supersonic flow.

To find the point of maximum entropy on Fanno line. From equation (5.19)

$$\frac{ds}{C_p} = \frac{1}{\gamma}\frac{dT}{T} - \frac{\gamma-1}{2\gamma}\frac{dT}{T_o-T} \qquad \div dT$$

$$\frac{ds}{dT} = \frac{1}{\gamma} C_p \frac{1}{T} - \frac{\gamma-1}{2\gamma} C_p \frac{1}{T_o-T} \qquad \left(\text{put } \frac{ds}{dT}=0\right)$$

$$\frac{1}{\gamma}\frac{1}{T} = \frac{\gamma-1}{2\gamma}\frac{1}{T_o-T} \qquad \because V^2 = 2C_p(T_o-T) \quad \text{and} \quad C_p = \frac{\gamma R}{\gamma-1}$$

$$\therefore \frac{1}{T} = \frac{\cancel{\gamma T}}{V^2}\,\frac{\gamma R}{\cancel{\gamma T}} \longrightarrow V^2 = \gamma R T \longrightarrow \boxed{M=1}$$

This means that M=1 at point of maximum entropy.
For Friction Flow and according to 2nd law A subsonic flow never becomes supersonic, and supersonic flow never becomes subsonic, unless a shock wave is present.

5.3. Relations For the Frictional Flow

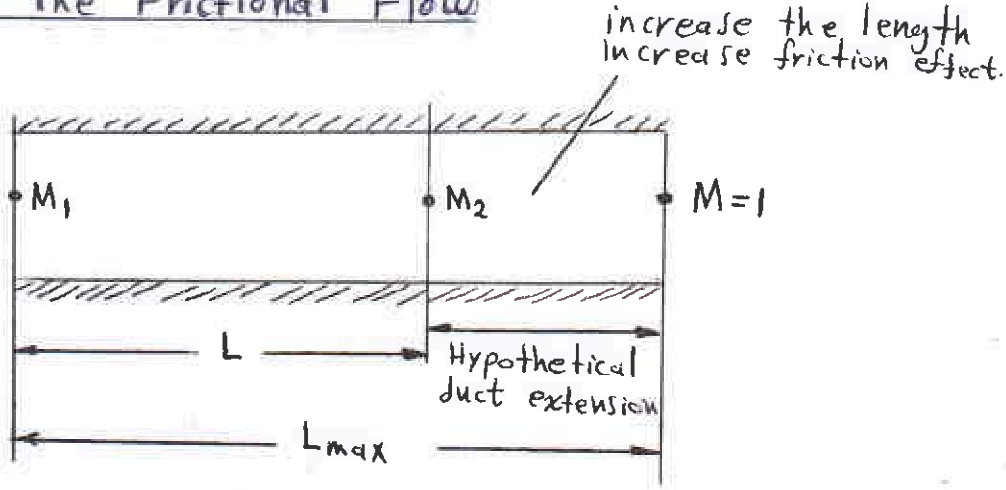


Figure (5.3)

By integrating equation (5.13) between a section where Mach number is (M) and the section where (M=1), we get:-

$$\int_{M^2}^{1} \frac{1-M^2}{\gamma M^4 \left(1+\frac{\gamma-1}{2}M^2\right)}\, dM^2 = \int_0^{L_{max}} 4f \frac{dx}{D} \quad \text{------} \quad (5.21)$$

Integration the left side by partial fraction.

$$\frac{4\bar{f} L_{max}}{D} = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma} \ln\left(\frac{(\gamma+1)M^2}{2\left(1+\frac{\gamma-1}{2}M^2\right)}\right) \quad \text{------} \quad (5.22)$$

where $\bar{f}$ is the mean friction factor, defined over the duct length as:-

$$\bar{f} = \frac{1}{L_{max}} \int_0^{L_{max}} f\, dx$$

The duct length (L) required for Mach number to change from M_1 to M_2 can be found as

$$\frac{4\bar{f}L}{D} = \left(\frac{4\bar{f}L_{max}}{D}\right)_{M_1} - \left(\frac{4\bar{f}L_{max}}{D}\right)_{M_2} \quad \text{------} \quad (5.23)$$

$$\because \frac{T}{T^*} = \frac{T}{T_o}\,\frac{T_o}{T^*} \qquad \frac{T_o}{T} = 1 + \frac{\gamma-1}{2}M^2, \text{ and } \frac{T_o}{T^*} = 1 + \frac{\gamma-1}{2} = \frac{\gamma+1}{2}$$

$$\boxed{\therefore \frac{T}{T^*} = \frac{\gamma+1}{2+(\gamma-1)M^2}} \quad \text{------- (5.24)}$$

$$\frac{V}{V^*} = \frac{M\sqrt{\gamma R T}}{\sqrt{\gamma R T^*}} = M\sqrt{\frac{T}{T^*}}$$

$$\boxed{\therefore \frac{V}{V^*} = M\sqrt{\frac{\gamma+1}{2+(\gamma-1)M^2}}} \quad \text{------- (5.25)}$$

By continuity equation for steady flow in constant area duct

$$\rho V = \rho^* V^*$$

$$\boxed{\frac{\rho}{\rho^*} = \frac{V^*}{V} = \frac{1}{M}\sqrt{\frac{2+(\gamma-1)M^2}{\gamma+1}}} \quad \text{------- (5.26)}$$

and by perfect gas law

$$\frac{p}{p^*} = \frac{\rho}{\rho^*}\,\frac{T}{T^*}$$

By using eqs. (5.24) and (5.26)

$$\boxed{\frac{p}{p^*} = \frac{1}{M}\sqrt{\frac{\gamma+1}{2+(\gamma-1)M^2}}} \quad \text{------- (5.27)}$$

$$\frac{P_o}{P_o^*} = \frac{P_o}{P}\,\frac{P}{P^*}\,\frac{P^*}{P_o^*}$$

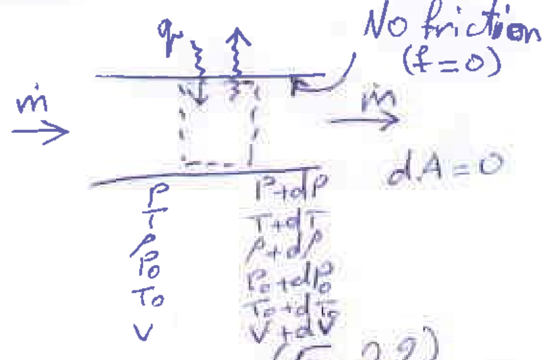
$$= \left[1+\frac{\gamma-1}{2}M^2\right]^{\frac{\gamma}{\gamma-1}} \quad \xrightarrow{5.27} \quad \frac{1}{M}\sqrt{\frac{\gamma+1}{2+(\gamma-1)M^2}} \qquad \frac{1}{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}}$$

$$= \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{-\frac{1}{2}}$$

$$\boxed{\frac{P_o}{P_o^*} = \frac{1}{M}\left(\frac{2+(\gamma-1)M^2}{(\gamma+1)}\right)^{\frac{\gamma+1}{2(\gamma-1)}}} \quad \text{------- (5.28)}$$

5.4. Frictionless Flow with Heat Transfer in Constant Area Duct

It is the flow with heat added or cooling with no friction. This is unreal hypothesis since it is impossible to neglect the friction except in very small duct length. Combustion chamber is a good example of this case.



Basic Equations

$dA = 0$ & $f = 0$

$$\therefore \frac{dT_o}{T_o} = \frac{1-M^2}{(1+\gamma M^2)(1+\frac{\gamma-1}{2}M^2)} \frac{dM^2}{M^2} \qquad (5.29)$$

$$\frac{dV}{V} = \frac{1}{1+\gamma M^2} \frac{dM^2}{M^2} \qquad (5.30)$$

$$\frac{dP}{P} = -\frac{\gamma M^2}{1+\gamma M^2} \frac{dM^2}{M^2} \qquad (5.31)$$

$$\frac{d\rho}{\rho} = -\frac{1}{1+\gamma M^2} \frac{dM^2}{M^2} \qquad (5.32)$$

$$\frac{dT}{T} = \frac{(1-\gamma M^2)(1-M^2)}{(1+\gamma M^2)} \frac{dM^2}{M^2} \qquad (5.33)$$

$$\frac{dP_o}{P_o} = -\frac{\gamma M^2(1-M^2)}{2(1+\gamma M^2)(1+\frac{\gamma-1}{2}M^2)} \frac{dM^2}{M^2} \qquad (5.34)$$

$$\frac{ds}{C_p} = \frac{1-M^2}{1+\gamma M^2} \frac{dM^2}{M^2} \qquad (5.35)$$

$$dq = C_p \, dT_o \qquad (5.36)$$

From Eq. (5.29) & Eq. (5.36),

at $M=1 \implies \frac{dq}{dM}=0$.

This means that the heat exchanged with the flow per unit of mass (q) reaches its maximum value. Therefor, there is no additional heat exchange for the same inlet conditions as the duct exit at the critical state. Hence, the critical state represents the state of the end point in the duct (exit).

By integrating Eq. (5.29) — Eq (5.35) between a certain point and critical state point, the following relations are obtained:

$$\frac{T_o}{T_o^*} = \frac{2(\gamma+1)M^2\left(1+\frac{\gamma-1}{2}M^2\right)}{(1+\gamma M^2)^2} \quad \text{——— (5.37)}$$

$$\frac{V}{V^*} = \frac{(\gamma+1)M^2}{1+\gamma M^2} \quad \text{——— (5.38)}$$

$$\frac{P}{P^*} = \frac{\gamma+1}{1+\gamma M^2} \quad \text{——— (5.39)}$$

$$\frac{\rho}{\rho^*} = \frac{1+\gamma M^2}{(\gamma+1)M^2} \quad \text{——— (5.40)}$$

$$\frac{T}{T^*} = \frac{(\gamma+1)^2 M^2}{(1+\gamma M^2)^2} \quad \text{——— (5.41)}$$

$$\frac{P_o}{P_o^*} = \frac{\gamma+1}{1+\gamma M^2}\left[\frac{2\left(1+\frac{\gamma-1}{2}M^2\right)}{\gamma+1}\right]^{\frac{\gamma}{\gamma-1}} \quad \text{——— (5.42)}$$

$$\frac{S-S^*}{C_p} = \ln\left[M^2\left(\frac{\gamma+1}{1+\gamma M^2}\right)^{\frac{\gamma}{\gamma-1}}\right] \quad \text{——— (5.43)}$$

Now, q between any two points in the duct can be calculated from

$$\boxed{q = C_p\,(T_{o2} - T_{o1})} \quad \text{——— (5.44)}$$

5.5 Rayleigh Line

It represents all points that can be reached from a starting point in a constant area duct that has a simple change in T_o. This line can be drawn in the T-S diagram using the folowing equation which is derived from Eq. (5.41) & Eq (5.43) as shown in Figure

$$\frac{S-S^*}{C_p} = \ln\left[\frac{T}{T^*}\left(\frac{1}{(\gamma+1)-\sqrt{(\gamma+1)^2-4\gamma\frac{T}{T^*}}}\right)^{\frac{\gamma-1}{\gamma}}\right] \quad (5.45)$$

Heat addition moves Mach number towards 1, while heat removal moves it away from 1.

Comparing with Fanno line, Rayleigh line has same behavior in heating but opposit result in cooling.

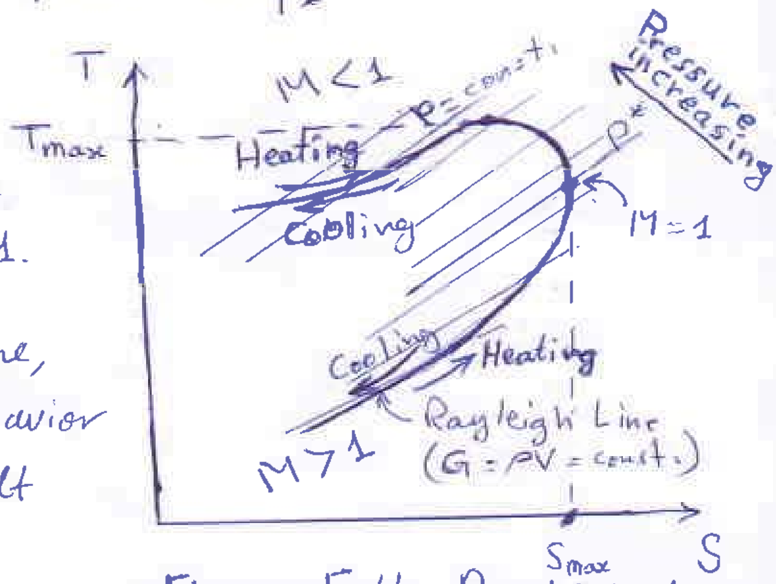


Figure 5.4. Rayleigh Line

For ease of calculations, Rayleigh flow characteristics of gas with $\gamma=1.4$ (such air) are listed in Rayleigh Flow table

Example (1)

Air flows in frictionless constant area duct from poin(1) at $V_1 = 100$ m/s, $P_1 = 50$ kPa and $T_1 = 50\,^\circ C$. If heat is added between point (1) and point (2) by a rate of 100 kca/Kg,

a) compute M_2, P_2, T_2, V_2 & P_{o2}

b) What is the maximum heat per unit of mass that can be added to the duct at same inlet conditions?

Solution

a)

① $$M_1 = \frac{V_1}{a_1} = \frac{V_1}{\sqrt{\gamma R T_1}} = \frac{100}{\sqrt{1.4 * 287 * 323}} = 0.278$$

② From isentropic flow table at $M_1 = 0.278$,

$$\frac{P_1}{P_{o1}} = 0.94772 \Longrightarrow P_{o1} = \frac{P_1}{0.94772} = \frac{50}{0.94772} = 52.8 \text{ kPa}$$

$$\frac{T_1}{T_{o1}} = 0.98477 \Longrightarrow T_{o1} = \frac{T_1}{0.98477} = \frac{323}{0.98477} = 328\ ^\circ K$$

③ Also, from Rayleigh flow table at $M_1 = 0.278$,

$$\frac{T_{o1}}{T_o^*} = 0.30674 \Longrightarrow T_o^* = \frac{T_{o1}}{0.30674} = \frac{328}{0.30674} = 1069.3\ ^\circ K$$

$$\frac{T_1}{T^*} = 0.36248 \Longrightarrow T^* = \frac{T_1}{0.36248} = \frac{323}{0.36248} = 891\ ^\circ K$$

$$\frac{P_{o1}}{P_o^*} = 1.2072 \Longrightarrow P_o^* = \frac{P_{o1}}{1.2072} = \frac{52.8}{1.2072} = 43.7 \text{ kPa}$$

$$\frac{P_1}{P^*} = 2.1656 \Longrightarrow P^* = \frac{P_1}{2.1656} = \frac{50}{2.1656} = 23 \text{ kPa}$$

$$\frac{V_1}{V^*} = 0.16739 \Longrightarrow V^* = \frac{V_1}{0.16739} = \frac{100}{0.16739} = 597.4 \text{ m/s}$$

Now, T_{02} can be calculated from Eq (5.44),

(4) $$T_{02} = T_{01} + \frac{q}{C_p} = 328 + \frac{100}{0.24} = 744\,^\circ K$$

$$\therefore \frac{T_{02}}{T_0^*} = \frac{744}{1069.3} = 0.6957$$

From Rayleigh table at $\frac{T_{02}}{T_0^*} = 0.695$

$M_2 = 0.503$ (subsonic) or $M_2 = 2.61$ (supersonic)

We select $\boxed{M_2 = 0.503}$ because $M_1 = 0.278$ (subsonic)

(5) $$\frac{T_2}{T^*} = 0.79389 \Rightarrow T_2 = T^* * 0.79389 = 891 * 0.79389 = \boxed{707\,^\circ K}$$

$$\frac{P_2}{P^*} = 1.7732 \Rightarrow P_2 = P^* * 1.7732 = 23 * 1.7732 = \boxed{40.8 \text{ kPa}}$$

$$\frac{P_{02}}{P_0^*} = 1.113 \Rightarrow P_{02} = P_0^* * 1.113 = 43.7 * 1.113 = \boxed{48.6 \text{ kPa}}$$

$$\frac{V_2}{V^*} = 0.44776 \Rightarrow V_2 = V^* * 0.44776 = 597.4 * 0.44776 = \boxed{267.5 \text{ m/s}}$$

b) $q = q_{max}$ at $T_{02} = T_0^*$

$$\therefore q_{max} = C_p (T_0^* - T_{01})$$

$$= 0.24 (1069.3 - 328) = \boxed{178 \text{ Kcal/kg}}$$