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1.1 Finite Element Method (FEM)

The finite element method is a numerical method for solving problems of engineering and mathematical physics.

1.1.1 Applications of the Finite Element Method

The finite element method can be used to analyze both structural and nonstructural problems. *Typical structural areas include*

1. Stress analysis, including truss and frame analysis, and stress concentration problems typically associated with holes, fillets, or other changes in geometry in a body.
2. Buckling.
3. Vibration analysis.

Nonstructural problems include

1. Heat transfer.
2. Fluid flow, including seepage through porous media.
3. Distribution of electric or magnetic potential.

1.1.2 Advantages of the Finite Element Method

This method has a number of advantages that have made it very popular. They include the ability to:

1. Model irregularly shaped bodies quite easily,
2. Handle general load conditions without difficulty,
3. Model bodies composed of several different materials because the element equations are evaluated individually,

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4. Handle unlimited numbers and kinds of boundary conditions,
5. Vary the size of the elements to make it possible to use small elements where necessary,
6. Alter the finite element model relatively easily and cheaply
7. Include dynamic effects
8. Handle nonlinear behavior existing with large deformations and nonlinear materials

1.1.3 General steps of the Finite Element Method

***Step* 1** Discretize and Select the Element Types.

***Step* 2** Select a Displacement Function.

***Step* 3** Define the Strain-Displacement and Stress-Strain Relationships

***Step* 4** Derive the Element Stiffness Matrix and Equations.

There are several to do that such as:

a. Direct Equilibrium Method.
b. Work or Energy Methods. Used for 2-D and 3-D elements
    - The virtual work method (elastic and inelastic materials)
    - The principle of minimum potential energy (elastic materials)
c. Methods of Weighted Residuals
    - Galerkin`s method.
    - The collocation method.
    - Least squares method.
    - Sub-domain method.

***Step* 5** Assemble the Element Equations to Obtain the Global or Total Equations and Introduce Boundary Conditions.

***Step* 6** Solve for the Unknown Degrees of Freedom (or Generalized Displacements).

***Step* 7** Solve for the Element Strains and Stresses.

***Step* 8** Interpret the Results.

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1.2 Definitions

The total potential energy (π_p): is defined as the sum of the internal strain (U) and the potential energy of the external forces (Ω), that is:

$$\pi_p = U + \Omega \qquad \textbf{...(1-1)}$$

Strain energy (U): is the capacity of internal forces (or stresses) to do work through the deformations (strains) in the structure.

Potential energy of external work (Ω): is the capacity of forces, *such as body forces, surface traction and nodal applied forces* to do work through the deformations of the structure.

The potential energy of structure is expressed in terms of displacements. In the finite element formulation, this will generally be nodal displacement such that:

$$\pi_p = \pi_p(d_1, d_2, d_3, \ldots\ldots d_n) \qquad \textbf{...(1-2)}$$

Where d_i are nodal displacements.

When π_p is minimized with respect to these displacements $\left(\dfrac{\partial \pi_p}{\partial d_i} = 0\right)$, the equilibrium equation ($kd = F$) result.

1.3 Work

1.3.1 Work done by force (W):

Let a force F moves a body from A to $\bar{A}$ of distance (s). The increment of work is:

$$dW = F.\,ds$$

and the total work is

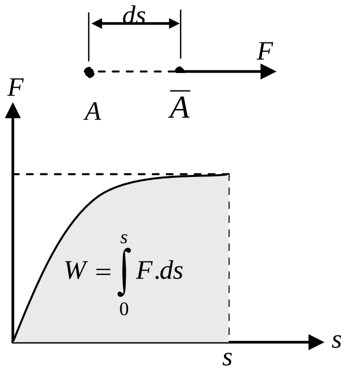

Fig.(1): Load-displacement relationship

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$$W = \int dW = \int_0^s F.ds = \text{ area under the curve}$$

If $\quad F = ks \quad$ linear as in spring

Then $\quad W = \int dW = \int_0^s F.ds = \int_0^s ks.ds = \frac{ks^2}{2} = \frac{1}{2}ks^2 = \frac{1}{2}(ks)s$

Thus $\quad W = \frac{1}{2}Fs \qquad$ (area of triangle) $\quad$ **…(1-3)**

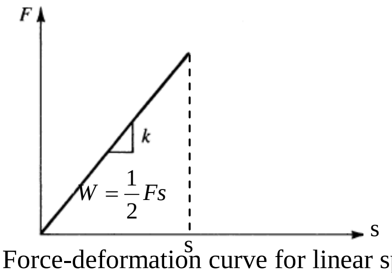

Fig. (2) Force-deformation curve for linear spring

1.3.2 Work done by a couple (or moment):

Let C a couple act on one end of a body of length l which pivoted at the other end as shown in Fig. (3). Let $d\phi$ be the rotation. Replace the couple C by two equal and opposite forces each equal $F = \frac{C}{l}$.

The increment of work is:

$$dW = \underbrace{F.ds}_{\text{Moving end}} + \underbrace{F * 0}_{\text{pivoted end}} = \frac{C}{l} \times l\, d\phi = C \times d\phi$$

so $\quad dW = C * d\phi$

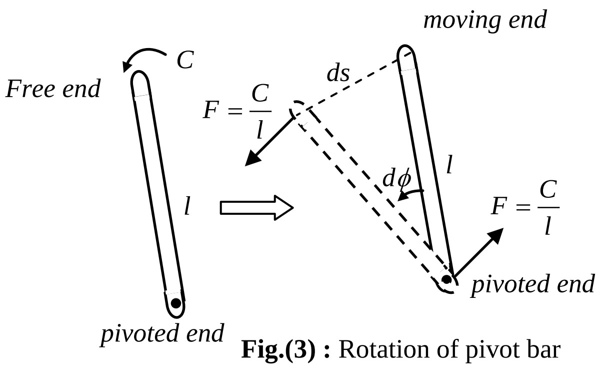

Fig.(3) : Rotation of pivot bar

and the total work is

$$W = \int dW = \int_0^\phi C.d\phi = \text{area under curve}$$

If $\quad C = k\phi \quad$ (linear rotational spring)

Then $\quad W = \int dw = \int_0^\phi C.d\phi = \int_0^\phi k\phi.d\phi = \frac{k\phi^2}{2} = \frac{1}{2}k\phi^2 = \frac{1}{2}(k\phi)\phi$

Thus $\quad W = \frac{1}{2}C\phi \qquad$ (area of triangle) $\qquad$ **…(1-4)**

***Note*:** The work done by external forces on a structure is stored in the structure as energy (strain energy).

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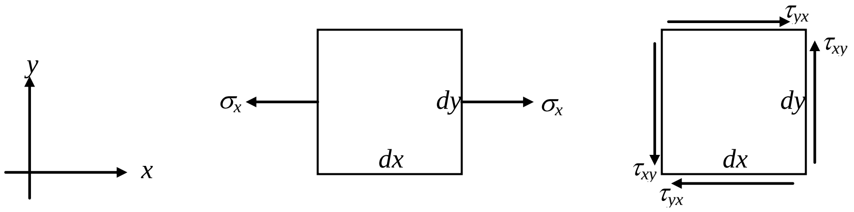
1.3.3 Work done by stresses in a body:

Consider a rectangular block ($dV = dx.dy.dz$) under normal stresses (σ_x, σ_y and σ_z) and under shear stresses($\tau_{xy} = \tau_{yx},\ \tau_{zy} = \tau_{yz}$ and $\tau_{zx} = \tau_{xz}$).

Note: Remember that normal stresses act twos but shearing stresses in fours to maintain equilibrium.

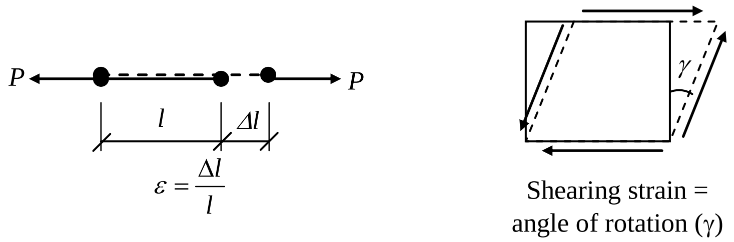


Typical element under Normal stresses

Typical element under shearing stresses

Fig. (4): Typical element under stresses maintaining equilibrium

Let the normal strains be (ε_x, ε_y and ε_z) and shearing strains (γ_{xy}, γ_{yz} and γ_{zx})



Typical element undergoes normal strain

Typical element undergoes shearing strain

Fig. (5): Typical element undergoes strain

To find the work (or energy) by the stresses in the block:

First assume σ_x is acting alone.

The work (or energy) by σ_x is:

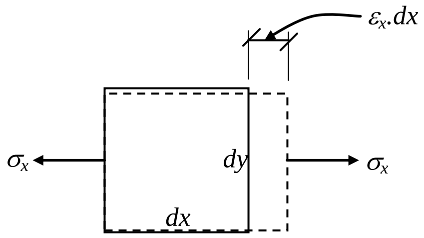


Fig. (6): Elongation due to normal stress

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$dU_1 = \frac{1}{2} F.ds = \frac{1}{2}(\sigma_x.dy\,dz) \times (\varepsilon_x.dx)$ where $dV = dx\ dy\ dz$

$So\ \ dU_1 = \frac{1}{2}\sigma_x\,\varepsilon_x.dV$

similarly

$dU_2 = \frac{1}{2}\sigma_y\,\varepsilon_y.dV$ and $dU_3 = \frac{1}{2}\sigma_z\,\varepsilon_z.dV$

Next the shearing stresses ($\tau_{xy} = \tau_{yx}$) are acting alone.

$dU_4 = \frac{1}{2} F.ds = \frac{1}{2}(\tau_{yx}.dx\,dz) \times (\gamma_{xy}.dy)$

$So\ \ dU_4 = \frac{1}{2}\tau_{yx}\,\gamma_{xy}.dV$

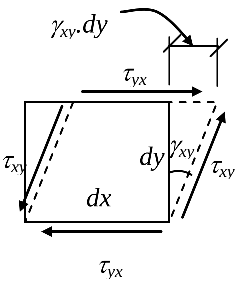

Fig. (7): Rotation due to shearing stress

similarly

$dU_5 = \frac{1}{2}\tau_{yz}\,\gamma_{yz}.dV$ and $dU_6 = \frac{1}{2}\tau_{zx}\,\gamma_{zx}.dV$

Now combine:

$dU = dU_1 + dU_2 + dU_3 + dU_4 + dU_5 + dU_6$

$$= \frac{1}{2}\ (\varepsilon_x\,\sigma_x + \varepsilon_y\,\sigma_y + \varepsilon_z\,\sigma_z + \gamma_{xy}\,\tau_{xy} + \gamma_{yz}\,\tau_{yz} + \gamma_{zx}\,\tau_{zx})dV \qquad \textbf{...(1-5a)}$$

Or in matrix form:

$$dU = \frac{1}{2}\begin{bmatrix}\varepsilon_x & \varepsilon_y & \varepsilon_z & \gamma_{xy} & \gamma_{yz} & \gamma_{zx}\end{bmatrix}\begin{bmatrix}\sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx}\end{bmatrix}dV \qquad \textbf{...(1-5b)}$$

or

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$$dU = \frac{1}{2}\begin{bmatrix}\sigma_x & \sigma_y & \sigma_z & \tau_{xy} & \tau_{yz} & \tau_{zx}\end{bmatrix}\begin{bmatrix}\varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx}\end{bmatrix} dV \qquad \text{...(1-5c)}$$

The total energy in the body is

$$U = \int_{vol.} dU = \int_{vol.} \frac{1}{2}\{\varepsilon\}^T \{\sigma\}.dV$$

$$= \int_{vol.} \frac{1}{2}\left(\varepsilon_x\sigma_x + \varepsilon_y\sigma_y + \varepsilon_z\sigma_z + \gamma_{xy}\tau_{xy} + \gamma_{yz}\tau_{yz} + \gamma_{zx}\tau_{zx}\right).dV \qquad \text{...(1-5d)}$$

1.4 Stress-Strain Relations:

1.4.1 Normal strain (changing length):

Take a line AB of length dx in x-direction. When it deforms, it goes to $A`B`$. So,

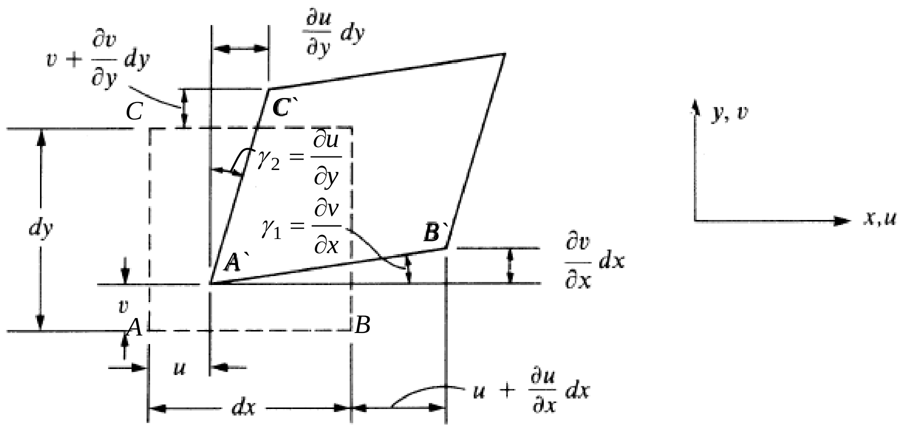

Fig. (8): Displacements and rotations of lines of an element in the xy-plane

$$Normal strain = \frac{changed length}{orginal length} = \frac{A`B` - AB}{AB}$$

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or $$\varepsilon_x = \frac{\left(dx + \frac{\partial u}{\partial x} dx\right) - dx}{dx} = \frac{\frac{\partial u}{\partial x} dx}{dx}$$

Thus $$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \textbf{...(1-6a)}$$

Similarly $$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \textbf{...(1-6b)}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \textbf{...(1-6c)}$$

where u, v and w are displacements along x, y and z-axes respectively.

1.4.2 Shear strain (changing angle)

Take two lines AB and AC at right angles in xy-plane. These will be $A`B`$ and $A`C`$ (see Fig.(8)):

$$shearstrain = \gamma_1 + \gamma_2 = \frac{\frac{\partial v}{\partial x} dx}{dx} + \frac{\frac{\partial u}{\partial y} dy}{dy}$$

$$\therefore \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad \textbf{...(1-7a)}$$

Similarly $$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \qquad \textbf{...(1-7b)}$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \qquad \textbf{...(1-7c)}$$

1.5 Hook's Law

1.5.1 Three-Dimensions Problems:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \upsilon\sigma_y - \upsilon\sigma_z\right] \qquad \textbf{...(1-8a)}$$

$$\varepsilon_y = \frac{1}{E}\left[-\upsilon\sigma_x + \sigma_y - \upsilon\sigma_z\right] \qquad \textbf{...(1-8b)}$$

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$$\varepsilon_z = \frac{1}{E}\left[-\upsilon\sigma_x - \upsilon\sigma_y + \sigma_z\right] \qquad \textbf{...(1-8c)}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\upsilon)\tau_{xy}}{E} \qquad \textbf{...(1-8d)}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} = \frac{2(1+\upsilon)\tau_{yz}}{E} \qquad \textbf{...(1-8e)}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} = \frac{2(1+\upsilon)\tau_{xz}}{E} \qquad \textbf{...(1-8f)}$$

Or in matrix form

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\upsilon & -\upsilon & 0 & 0 & 0 \\ -\upsilon & 1 & -\upsilon & 0 & 0 & 0 \\ -\upsilon & -\upsilon & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\upsilon) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\upsilon) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\upsilon) \end{bmatrix}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \qquad \textbf{...(1-8g)}$$

Where E: modulus of elasticity,

υ: Poisson's ratio

And the inverse relations:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\upsilon)(1-2\upsilon)}\begin{bmatrix} 1-\upsilon & \upsilon & \upsilon & 0 & 0 & 0 \\ \upsilon & 1-\upsilon & \upsilon & 0 & 0 & 0 \\ \upsilon & \upsilon & 1-\upsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\upsilon)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\upsilon)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\upsilon)/2 \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \qquad \textbf{...(1-9)}$$

Where $G = \dfrac{E}{2(1+\upsilon)}$

Note: If $\upsilon \to 0.5$, then $\sigma_x, \sigma_y \; or \; \sigma_z \to \infty$ (incompressible material)

1.5.2 Two-Dimensions Problems:

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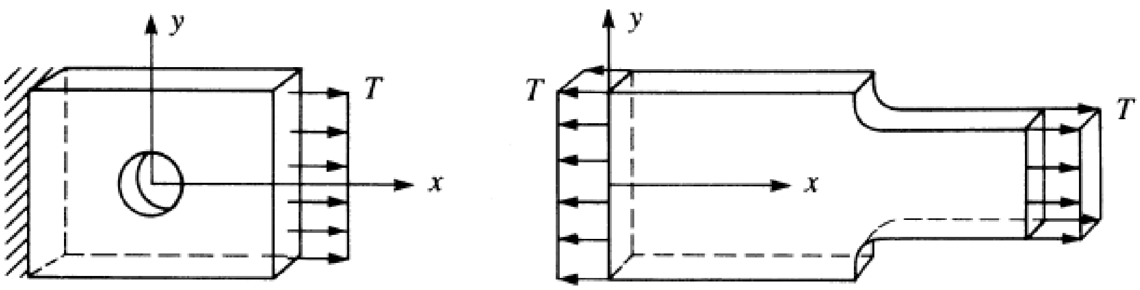
There are two types of two-dimensional problems

1.5.2.1 Plane stress:

Plane stress is defined to be a state of stress in which the normal stress and the shear stresses directed perpendicular to the plane are assumed to be zero, that is

$$\sigma_z = \tau_{yz} = \tau_{xz} = 0 \qquad (\varepsilon_z \neq 0) \qquad \textbf{...(1-10)}$$



(a) plate with hole **(b)** plate with fillet

Fig.(9): Plane stress problems

Generally, members that are thin (those with a small z dimension compared to the in-plane x and y dimensions) and whose loads act only in the xy-plane can be considered to be under plane stress.

1.5.2.2 Plane Strain

Plane strain is defined to be a state of strain in which the strain normal to the xy-plane ε_z and the shear strains γ_{xz} and γ_{yz} are assumed to be zero. That is:

$$\varepsilon_z = \gamma_{yz} = \gamma_{xz} = 0 \qquad (\sigma_z \neq 0) \qquad \textbf{...(1-11)}$$

The assumptions of plane strain are realistic for long bodies (say, in the z direction) with constant cross-sectional area subjected to loads that act only in the x and/or y directions and do not vary in the z direction.

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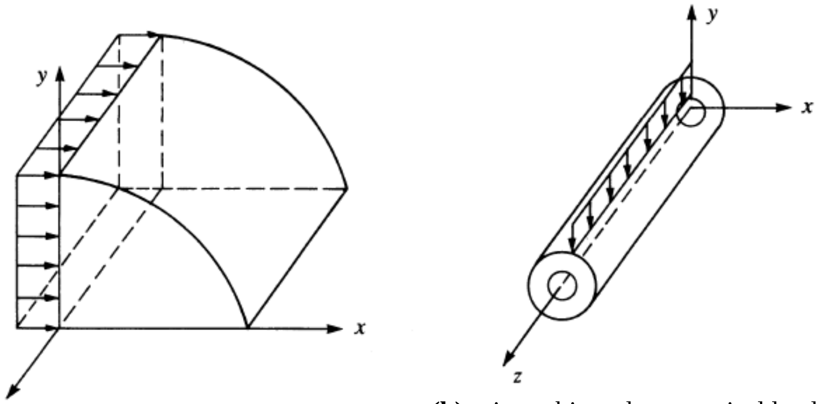


(a): dam subjected to horizontal loading **(b):** pipe subjected to a vertical load

Fig.(10): Plane strain problems:

The two dimensional state of stress and strain is:

$$\{\sigma\}=[D]\{\varepsilon\} \qquad \textbf{...(1-12)}$$

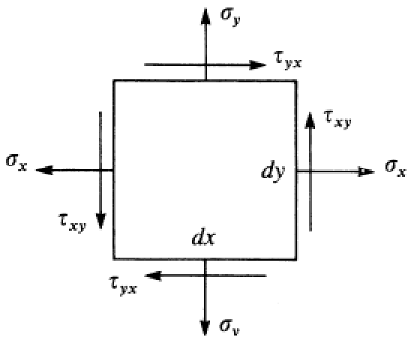


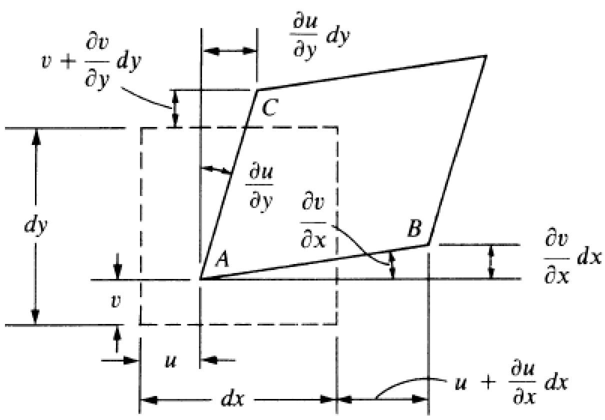
Fig.(11): Two-dimensional state of stress

Where $\{\sigma\}=\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}$; $\qquad \{\varepsilon\}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$

While $[D]_{3\times3}$ has two definitions

- ***Plane stress***

$$[D]=\frac{E}{1-\upsilon^2}\begin{bmatrix}1 & \upsilon & 0\\ \upsilon & 1 & 0\\ 0 & 0 & \frac{1-\upsilon}{2}\end{bmatrix} \qquad \textbf{...(1-13)}$$



Where $\quad E$: modulus of elasticity,

υ: Poisson's ratio

- ***Plane strain***

From Eq. 1-9

Fig.(12): Displacements and rotations of lines of an element in the xy-plane

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$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\upsilon)(1-2\upsilon)} \begin{bmatrix} 1-\upsilon & \upsilon & 0 \\ \upsilon & 1-\upsilon & 0 \\ 0 & 0 & (1-2\upsilon)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \textbf{...(1-13a)}$$

$\sigma_z = \frac{E}{(1+\upsilon)(1-2\upsilon)}(\upsilon\varepsilon_x + \upsilon\varepsilon_y)$ is not zero; $\tau_{yz} = \tau_{xz} = 0$

or in the other words

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E`}{1-\upsilon`^2} \begin{bmatrix} 1 & \upsilon` & 0 \\ \upsilon` & 1 & 0 \\ 0 & 0 & (1-\upsilon`)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \textbf{...(1-13b)}$$

Thus $$[D] = \frac{E`}{1-\upsilon`^2} \begin{bmatrix} 1 & \upsilon` & 0 \\ \upsilon` & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon`}{2} \end{bmatrix} \quad \textbf{...(1-14)}$$

Where $E` = \frac{E}{1-\upsilon^2}$; $\upsilon` = \frac{\upsilon}{1-\upsilon}$ and $G` = G$

And the strains are:

$$\varepsilon_x = \frac{1}{E`}(\sigma_x - \upsilon`\sigma_y) \quad \textbf{...(1-15a)}$$

$$\varepsilon_y = \frac{1}{E`}(-\upsilon`\sigma_x + \sigma_y) \quad \textbf{...(1-15b)}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G`} = \frac{\tau_{xy}}{G} = \frac{2(1+\upsilon)\tau_{xy}}{E} \quad \textbf{...(1-15c)}$$