Spring elements

2.1 Derivation of the Stiffness Matrix for a Spring Element

***Step* 1** Discretize and Select the Element Types.

Here, the spring element has 2 nodes and there is one degree of freedom (*d.o.f*) in each node.

The element has (2*1=2 *d.o.f*)

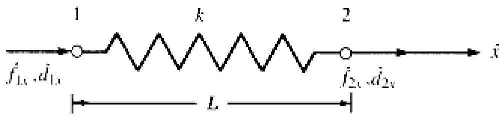

Fig.(2-1): Linear spring element with positive nodal displacement and force conventions.

Note: Forces and displacements are positive if they are in the positive direction of coordinate.

***Step* 2** Select a Displacement Function.

Use polynomials with total number of coefficients (a_i) are equal to the number of (*d.o.f*) associated with the element.

$$\hat{u} = a_1 + a_2\hat{x} \qquad \text{...(2-1a)}$$

or in matrix form: $$\{\hat{u}\} = [1 \quad \hat{x}]\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \qquad \text{...(2-1b)}$$

Find a_1 and a_2 by applying boundary conditions

at ***node* 1** $\quad \hat{x} = 0 \quad \Rightarrow \quad u = \hat{d}_{1x} \quad \Rightarrow \quad \hat{d}_{1x} = a_1$

at ***node* 2** $\quad \hat{x} = L \quad \Rightarrow \quad u = \hat{d}_{2x} \quad \Rightarrow \quad \hat{d}_{2x} = a_1 + a_2 L$

$$\because \; a_1 = \hat{d}_{1x} \quad \Rightarrow \quad \hat{d}_{2x} = \hat{d}_{1x} + a_2 L \quad \Rightarrow \quad a_2 = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}$$

Then $$\hat{u} = \hat{d}_{1x} + \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}x \qquad \text{...(2-2a)}$$

Rewrite as: $$\hat{u} = \left(1 - \frac{\hat{x}}{L}\right)\hat{d}_{1x} + \frac{\hat{x}}{L}\hat{d}_{2x} \qquad \text{...(2-2b)}$$

$$\hat{u} = [N_1 \quad N_2]\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix} \qquad \text{...(2-2c)}$$

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Where $N_1 = 1 - \frac{\hat{x}}{L}$ and $N_2 = \frac{\hat{x}}{L}$ **…(2-2d)**

Note: Notice that $N_1 = 1$ at node 1 and $N_1 = 0$ at node 2.

And also $N_2 = 0$ at node 1 and $N_2 = 1$ at node 2.

So that $N_1 + N_2 = 1$ at any axial coordinate along the element.

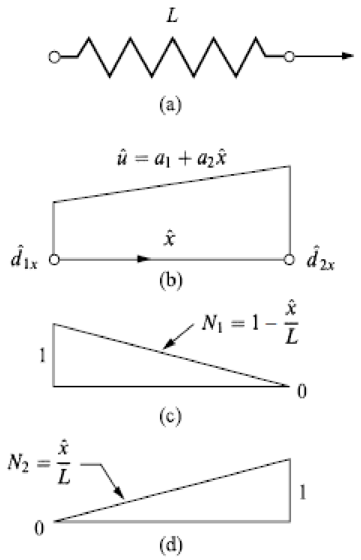

Fig.(2-2): (a) Spring element showing plots of (b) displacement function $\hat{u}$ and shape functions (c) N_1 and (d) N_2 over domain of element

***Step* 3** Define the Strain-Displacement and Stress-Strain Relationships

For a spring element, we can relate the force in the spring directly to the deformation. Therefore, the strain/displacement relationship is not necessary here.

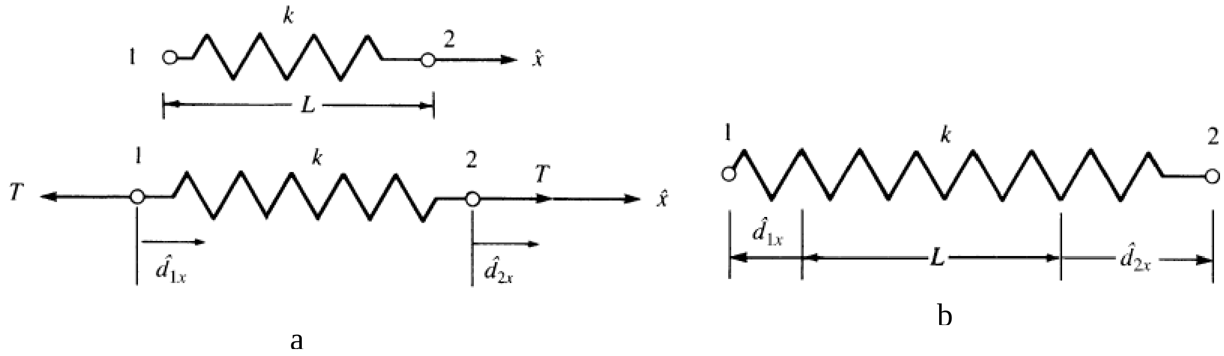

Fig.(2-3): a: Linear spring subjected to tensile forces b: deformed shape

The stress/strain relationship can be expressed in terms of the force/deformation relationship instead as:

$$T = k\delta \qquad \Rightarrow \qquad T = k(\hat{d}_{2x} - \hat{d}_{1x}) \qquad \textbf{…(2-3)}$$

***Step* 4** Derive the Element Stiffness Matrix and Equations.

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Direct equilibrium method will be used in this derivation.

***Case* a:** Assume that only node 1 can deflect and node 2 being fixed.

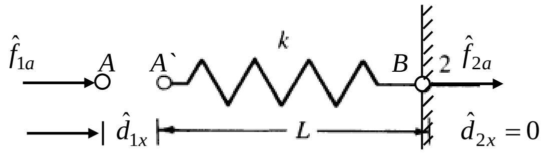

Fig. (2-4a): Case 1: Force $\hat{f}_{1a}$ is applied at node 1 and node 2 is fixed

$$\hat{f}_{1a} = k\hat{d}_{1x}$$

And from equilibrium ($\Sigma F_x = 0$)

$$\hat{f}_{1a} + \hat{f}_{2a} = 0$$

$$k\hat{d}_{1x} + \hat{f}_{2a} = 0$$

$$\therefore \quad \hat{f}_{2a} = -k\hat{d}_{1x} \qquad \textbf{...(2-4a)}$$

***Case* b:** Assume that only node 2 can deflect and node 1 being fixed.

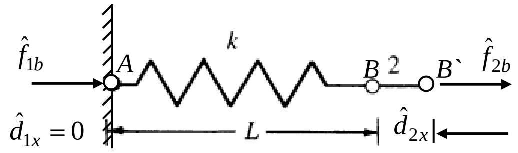

Fig. (2-4b): Case 2: Force $\hat{f}_{2b}$ is applied at node 2 and node 1 is fixed

$$\hat{f}_{2b} = k\hat{d}_{2x}$$

and from equilibrium ($\Sigma F_x = 0$)

$$\hat{f}_{1b} + \hat{f}_{2b} = 0$$

$$\hat{f}_{1b} + k\hat{d}_{2x} = 0$$

$$\therefore \quad \hat{f}_{1b} = -k\hat{d}_{2x} \qquad \textbf{...(2-4b)}$$

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Using the principle of superposition by combining the load systems:

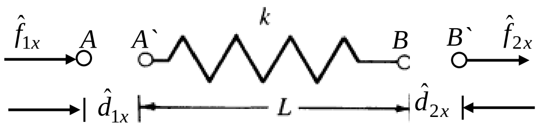

Fig. (2-4c): The principle of superposition (combination of case 1 & case 2)

The total forces acting at node 1:

$$\hat{f}_{1x} = \hat{f}_{1a} + \hat{f}_{1b} = k\hat{d}_{1x} - k\hat{d}_{2x} \qquad \textbf{...(2-5a)}$$

$$\hat{f}_{2x} = \hat{f}_{2a} + \hat{f}_{2b} = -k\hat{d}_{1x} + k\hat{d}_{2x} \qquad \textbf{...(2-5b)}$$

In matrix form $\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$ **...(2-6)**

Thus the stiffness matrix $[k^e]$ for single spring is:

$$[\hat{k}^e] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \qquad \textbf{...(2-7)}$$

Where $[\hat{k}^e]$ matrix is called local stiffness matrix.

***Step* 5** Assemble the Element Equations to Obtain the Global or Total Equations and Introduce Boundary Conditions.

This step applies for structures composed of more than one element such that: $[K] = \sum_{e=1}^{n} [\hat{k}^e]$ and $\{F\} = \sum_{e=1}^{n} \{\hat{f}^e\}$ **...(2-8)**

Where n is the total number of elements.

***Step* 6** Solve for the Unknown Degrees of Freedom (or Generalized Displacements).

The displacements are then determined by imposing boundary conditions, such as support conditions, and solving a system of equations, $\{F\} = [K]\{d\}$, simultaneously.

***Step* 7** Solve for the Element Strains and Stresses.

Finally, the element forces are determined by back-substitution, applied to each element, into equations similar to Eqs. (2-5).

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2.2 Stiffness Matrix for Assembly Spring

The stiffness matrix [K] for the collinear spring shown in Fig. (2-5) can be derived as follows:

The x axis is the global axis of the assemblage. The local $\hat{x}$ axis of each element coincides with the global axis of the assemblage.

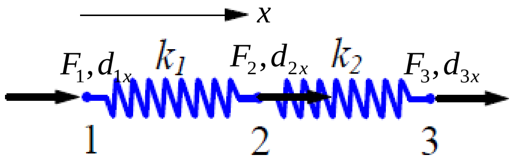

Fig.(2-5): Collinear spring system

Case a: Put $d_{2x} = d_{3x} = 0$ so $f_{1a} = k_1 d_{1x}$

Note that no force exists at node 3, since d_{2x} and d_{3x} are specified as zero.

$$f_{3a} = 0$$

And from equilibrium ($\Sigma F_x = 0$)

$$f_{1a} + f_{2a} + \cancelto{0}{f_{3a}} = 0$$

$$k_1 d_{1x} + f_{2a} + 0 = 0 \qquad \Rightarrow \qquad f_{2a} = -k_1 d_{1x}$$

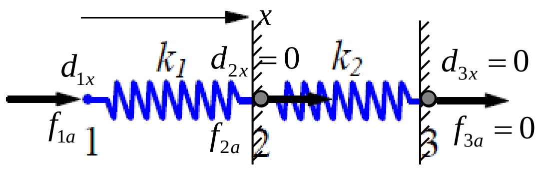

Fig.(2-6a): Case1 : node 1 is free and node 2 and 3 are fixed

Case 2: Put $d_{1x} = d_{3x} = 0$.

It can be noted that in this case continuity displacement at node 2 require that each spring deflect the same amount.

Thus the force at node 2 consists of two components ($k_1 d_{2x}, k_2 d_{2x}$)

So $\quad f_{2b} = (k_1 + k_2) d_{2x}$

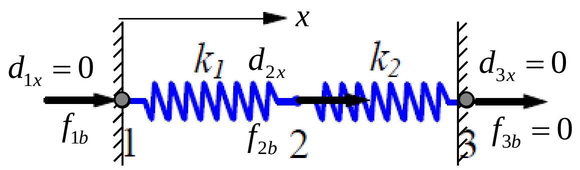

Fig.(2-6b): Case2 : node 2 is free and node 1 & 3 are fixed

Consider the equilibrium of each spring individually.

***Note*:** From compatibility condition we can conclude that $d_{2b}^{e_1} = d_{2b}^{e_2} = d_{2b}$.

Element 1

$$f_{1b} = -k_1 d_{2b}^{e_1} = -k_1 d_{2b}$$

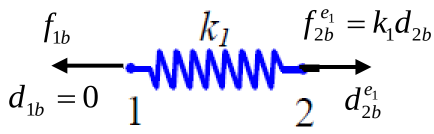

Fig.(2-6c): internal forces and displacements in element 1

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Element 2

$$f_{3b} = -k_2 d_{2b}^{e_2} = -k_2 d_{2b}$$

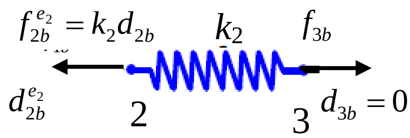

Fig.(2-6d): internal forces and displacements in element 2

Case c: Put $d_{1x} = d_{2x} = 0$ so $\boxed{f_{3c} = k_2 d_{3x}}$

Note that no force exists at node 1, since d_{1x} and d_{2x} are specified as zero.

$$f_{1c} = 0$$

And from equilibrium ($\Sigma F_x = 0$)

$$\underbrace{\cancel{f_{1c}}}_{0} + f_{2c} + f_{3c} = 0$$

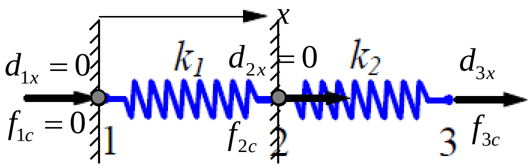

Fig.(2-6e): Case3 : node 3 is free and node 1 and 2 are fixed

$$0 + f_{2c} + k_2 d_{3x} = 0 \qquad \Rightarrow \qquad f_{2c} = -k_2 d_{3x}$$

By superposition:

		CASE 1	CASE 2	CASE 3
The total force acting at node 1	F_1=	$k_1 d_{1x}$	$-k_1 d_{2x}$	0
The total force acting at node 2	F_2=	$-k_1 d_{1x}$	$k_1 d_{2x} + k_2 d_{2x}$	$-k_2 d_{3x}$
The total force acting at node 3	F_3=	0	$-k_2 d_{2x}$	$k_2 d_{3x}$

Writing these equations in matrix form gives:

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \qquad \text{...(2-9)}$$

Note: The stiffness matrix $[K]$ is symmetric.

Note: There is another more convenient method for constructing the total stiffness matrix as it will be indicated.

First $[k^e]$ of the constituent element is written down

For Element 1 $\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$

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for Element 2 $$\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}$$

by inserting rows and columns of zeros, both may be expanded in such way that they relate to the three displacements (d_{1x}, d_{2x} and d_{3x}) thus,

For Element 1 $$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix}$$

For Element 2 $$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix}$$

The rule for matrix addition may be used to obtain,

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix}$$

Then apply the boundary conditions and eliminate the columns and rows corresponding to the zero displacements and solve the resulted equations for the displacements. Then, if required, put the calculated displacement on the original equations to find the reactions. Finally, use the force-displacements relations of the element to obtain the internal element forces.

2.3 Properties of the Stiffness Matrix

1. $[K]$ is symmetric, as is each of the element stiffness matrices.
2. $[K]$ is singular, and thus no inverse exists until sufficient boundary conditions are imposed to remove the singularity and prevent rigid body motion.
3. The main diagonal terms of $[K]$ are always positive. Otherwise, a positive nodal force F_i could produce a negative displacement d_i (a behavior contrary to the physical behavior of any actual structure).

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Example 2.1: For the spring assemblage with arbitrarily numbered nodes shown in Fig.(2–7), obtain (a) the global stiffness matrix, (b) the displacements of nodes 3 and 4, (c) the reaction forces at nodes 1 and 2, and (d) the forces in each spring. A force of 5000 lb is applied at node 4 in the x direction. The spring constants are given in the figure. Nodes 1 and 2 are fixed.

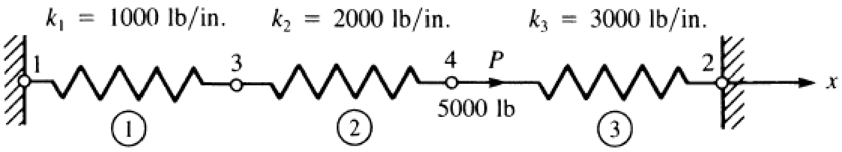

Fig. (2–7): Spring assemblage for solution

Solution:

1. The stiffness matrix of each element is,

$$[k^{(1)}]=\begin{array}{c}\begin{array}{cc}1 & 3\end{array}\\ \begin{bmatrix}1000 & -1000\\ -1000 & 1000\end{bmatrix}\end{array}\begin{array}{c}1\\3\end{array}\qquad [k^{(2)}]=\begin{array}{c}\begin{array}{cc}3 & 4\end{array}\\ \begin{bmatrix}2000 & -2000\\ -2000 & 2000\end{bmatrix}\end{array}\begin{array}{c}3\\4\end{array}$$

$$[k^{(3)}]=\begin{array}{c}\begin{array}{cc}4 & 2\end{array}\\ \begin{bmatrix}3000 & -3000\\ -3000 & 3000\end{bmatrix}\end{array}\begin{array}{c}4\\2\end{array}$$

2. Then expand each matrix with dimensions equal the dimensions of the stiffness matrix of the whole structure.

$$[k^{(1)}]=\begin{array}{c}\begin{array}{cccc}1 & 2 & 3 & 4\end{array}\\ \begin{bmatrix}1000 & 0 & -1000 & 0\\ 0 & 0 & 0 & 0\\ -1000 & 0 & 1000 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{array}\begin{array}{c}1\\2\\3\\4\end{array}\qquad [k^{(2)}]=\begin{array}{c}\begin{array}{cccc}1 & 2 & 3 & 4\end{array}\\ \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 2000 & -2000\\ 0 & 0 & -2000 & 2000\end{bmatrix}\end{array}\begin{array}{c}1\\2\\3\\4\end{array}$$

$$[k^{(3)}]=\begin{array}{c}\begin{array}{cccc}1 & 2 & 3 & 4\end{array}\\ \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 3000 & 0 & -3000\\ 0 & 0 & 0 & 0\\ 0 & -3000 & 0 & 3000\end{bmatrix}\end{array}\begin{array}{c}1\\2\\3\\4\end{array}$$

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3. Obtain the global stiffness matrix: $[K]=[k^{(1)}]+[k^{(2)}]+[k^{(3)}]$

$$[K]=\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 1000+2000 & -2000 \\ 0 & -3000 & -2000 & 2000+3000 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

4. The global stiffness matrix relates the global forces to global displacement ($\{F\}=[K]\{d\}$) as follows,

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 3000 & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \\ d_{4x} \end{Bmatrix}$$

5. Apply the boundary conditions,

$$\begin{Bmatrix} R_1 \\ R_2 \\ 0 \\ 5000 \end{Bmatrix} = \begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 3000 & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ d_{3x} \\ d_{4x} \end{Bmatrix}$$

6. Eliminate the rows and columns corresponding to zero displacements yields,

$$\begin{Bmatrix} R_1 \\ R_2 \\ 0 \\ 5000 \end{Bmatrix} = \begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 3000 & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ d_{3x} \\ d_{4x} \end{Bmatrix}$$

Thus $$\begin{Bmatrix} 0 \\ 5000 \end{Bmatrix} = \begin{bmatrix} 3000 & -2000 \\ -2000 & 5000 \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{4x} \end{Bmatrix}$$

7. Solve the above simultaneous equations yields,

$$d_{3x}=\frac{10}{11}in \quad ; \quad d_{4x}=\frac{15}{11}in$$

8. To obtain the global nodal forces (which include the reactions at nodes 1 and 2), we back-substitute. This substitution yields,

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$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 3000 & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 10/11 \\ 15/11 \end{Bmatrix}$$

Multiplying above matrices and simplifying, we obtain the forces at each node,

$$F_{1x} = \frac{-10000}{11} lb; \qquad F_{2x} = \frac{-45000}{11} lb; \qquad F_{3x} = 0 lb; \qquad F_{4x} = \frac{55000}{11} lb = 5000 lb$$

From these results, you can check the equilibrium conditions.

9. Next we use local element to obtain the forces in each element (internal forces).

Element 1 (nodes 1-3) $\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ 10/11 \end{Bmatrix}$

Simplifying, $\hat{f}_{1x} = \dfrac{-10000}{11} lb; \qquad \hat{f}_{3x} = \dfrac{10000}{11} lb$

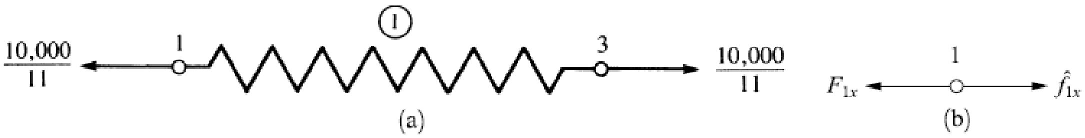

Fig. (2-8): (a) Free-body diagram of element 1 and (b) free-body diagram of node 1.

Element 2 (nodes 3-4) $\begin{Bmatrix} \hat{f}_{3x} \\ \hat{f}_{4x} \end{Bmatrix} = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix} \begin{Bmatrix} 10/11 \\ 15/11 \end{Bmatrix}$

Simplifying, $\hat{f}_{3x} = \dfrac{-10000}{11} lb; \qquad \hat{f}_{4x} = \dfrac{10000}{11} lb$

Fig. (2-9): Free-body diagram of element 2.

Element 3 (nodes 4-2) $\begin{Bmatrix} \hat{f}_{4x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix} \begin{Bmatrix} 15/11 \\ 0 \end{Bmatrix}$

Simplifying, $\hat{f}_{4x} = \dfrac{45000}{11} lb; \qquad \hat{f}_{2x} = \dfrac{-45000}{11} lb$

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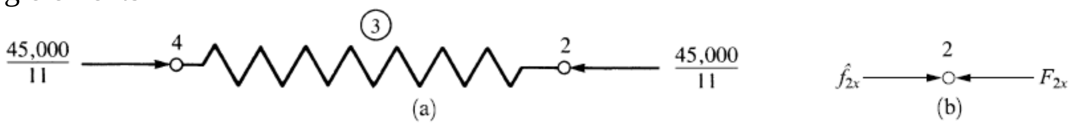

Fig. (2-10): (a) Free-body diagram of element 1 and (b) free-body diagram of node 2.

Example 2.2: For the spring assemblage shown in Fig. (2–11), obtain (a) the global stiffness matrix, (b) the displacements of nodes 2–4, (c) the global nodal forces, and (d) the local element forces. Node 1 is fixed while node 5 is given a fixed, known displacement δ=20 *mm*. The spring constants are all equal to k = 200 *kN/m*.

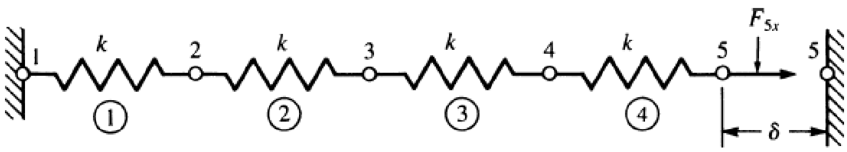

Fig. (2-11): Spring assemblage for solution

Solution:

1. The stiffness matrix of each element is

$$[k^1]=[k^2]=[k^3]=[k^4]=\begin{array}{c} \begin{array}{cc} 4 & 5 \\ 3 & 4 \\ 2 & 3 \\ 1 & 2 \end{array} \\ \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \end{array} \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{array}$$

2. So, the global stiffness matrix

$$[K]_{5\times5}=\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 200 & -200 & 0 & 0 & 0 \\ -200 & 400 & -200 & 0 & 0 \\ 0 & -200 & 400 & -200 & 0 \\ 0 & 0 & -200 & 400 & -200 \\ 0 & 0 & 0 & -200 & 200 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} kN/m$$

3. The global stiffness matrix relates the global forces to the global displacements as follows:

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$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \\ F_{5x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 & 0 & 0 & 0 \\ -200 & 400 & -200 & 0 & 0 \\ 0 & -200 & 400 & -200 & 0 \\ 0 & 0 & -200 & 400 & -200 \\ 0 & 0 & 0 & -200 & 200 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \\ d_{4x} \\ d_{5x} \end{Bmatrix}$$

4. Applying the boundary conditions $d_{1x}= 0$ and $d_{5x}=20\ mm$ (= 0.02 m), substituting known global forces $F_{2x}= 0$, $F_{3x}= 0$, and $F_{4x}= 0$, and partitioning the first and fifth of above equations corresponding to these boundary conditions, we obtain,

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} -200 & 400 & -200 & 0 & 0 \\ 0 & -200 & 400 & -200 & 0 \\ 0 & 0 & -200 & 400 & -200 \end{bmatrix} \begin{Bmatrix} 0 \\ d_{2x} \\ d_{3x} \\ d_{4x} \\ 0.02m \end{Bmatrix}$$

We now rewrite above equations transposing the product of the appropriate stiffness coefficient (200) multiplied by the known displacement (0.02 m) to the left side.

$$\begin{Bmatrix} 0 \\ 0 \\ 4kN \end{Bmatrix} = \begin{bmatrix} 400 & -200 & 0 \\ -200 & 400 & -200 \\ 0 & -200 & 400 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \\ d_{4x} \end{Bmatrix}$$

Solving, we obtain $d_{2x} = 0.005\ m$, $d_{3x} = 0.01\ m$, and $d_{4x} = 0.015\ m$.

5. The global forces obtained by back-substitution the boundary conditions.

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \\ F_{5x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 & 0 & 0 & 0 \\ -200 & 400 & -200 & 0 & 0 \\ 0 & -200 & 400 & -200 & 0 \\ 0 & 0 & -200 & 400 & -200 \\ 0 & 0 & 0 & -200 & 200 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.005 \\ 0.010 \\ 0.015 \\ 0.020 \end{Bmatrix}$$

$\therefore F_{1x} = -1.0kN$; $\quad F_{2x} = 0$; $\quad F_{3x} = 0$; $\quad F_{4x} = 0$ and $\quad F_{5x} = 1.0kN$

6. We make use of local element equation to obtain the forces in each element.

Element 1 $\quad \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.005 \end{Bmatrix}$

Simplifying yields

$$\hat{f}_{1x} = -1.0\,kN\,; \qquad \hat{f}_{2x} = 1.0\,kN$$

Element 2 $$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.005 \\ 0.010 \end{Bmatrix}$$

Simplifying yields

$$\hat{f}_{2x} = -1.0\,kN\,; \qquad \hat{f}_{3x} = 1.0\,kN$$

Element 3 $$\begin{Bmatrix} \hat{f}_{3x} \\ \hat{f}_{4x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.010 \\ 0.015 \end{Bmatrix}$$

Simplifying yields

$$\hat{f}_{3x} = -1.0\,kN\,; \qquad \hat{f}_{4x} = 1.0\,kN$$

Element 4 $$\begin{Bmatrix} \hat{f}_{4x} \\ \hat{f}_{5x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.015 \\ 0.020 \end{Bmatrix}$$

Simplifying yields

$$\hat{f}_{4x} = -1.0\,kN\,; \qquad \hat{f}_{5x} = 1.0\,kN$$

Example 2.3: Formulate the global stiffness matrix for the system of linear springs shown below, and then find the unknown displacements and reactions.

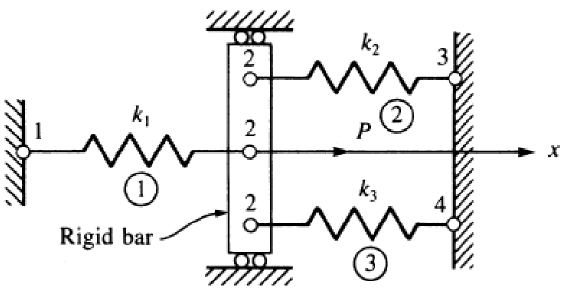

Fig. (2–12): Spring assemblage for solution

Solution:

1. The stiffness matrix of each element is,

$$[k^{(1)}] = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} \qquad [k^{(2)}] = \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \end{array}$$

Spring elements

$$[k^{(3)}] = \begin{array}{c c} \begin{array}{cc} 2 & 4 \end{array} & \\ \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} & \begin{array}{c} 2 \\ 4 \end{array} \end{array}$$

2. The global stiffness matrix is,

$$[K] = \begin{array}{c c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

3. Applying the boundary conditions ($d_{1x}= 0$, $d_{3x}= 0$, $d_{4x}= 0$ and $F_{2x}= P$),

$$\begin{Bmatrix} F_{1x} \\ P \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} 0 \\ d_{2x} \\ 0 \\ 0 \end{Bmatrix}$$

4. Eliminating the rows and columns corresponding to zero displacement

$$\begin{Bmatrix} F_{1x} \\ P \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} 0 \\ d_{2x} \\ 0 \\ 0 \end{Bmatrix}$$

Solving, we obtain

$$\{P\} = [k_1 + k_2 + k_3]\{d_{2x}\} \quad \therefore d_{2x} = \frac{P}{k_1 + k_2 + k_3}$$

5. The reactions could be found by back substitution

$$\begin{Bmatrix} R_1 \\ P \\ R_3 \\ R_4 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} 0 \\ P/(k_1+k_2+k_3) \\ 0 \\ 0 \end{Bmatrix}$$

$$\therefore R_1 = \frac{-k_1}{k_1+k_2+k_3} P; \qquad R_3 = \frac{-k_2}{k_1+k_2+k_3} P \qquad R_4 = \frac{-k_3}{k_1+k_2+k_3} P$$