First Order Ordinary Differential Equation (ODE)

2- Linear Equation

If the given 1st ODE can be written as:

$$\frac{dy}{dx} + p(x)y = Q(x)$$

It is called **linear equation in y**.

p(x) & Q(x) are function of x only or may be constants this form called **standard form**.

The integrating factor (I.F) $\mu(x) = e^{\int p(x)dx}$

And the solution is: $\mu(x).y = \int Q(x).\mu(x)\,dx + c$

OR

by integrating both sides with respect to x, gives: $\mu(x).\ y = \int Q(x).\mu(x)\,dx + c$

Ex: Solve $\frac{dy}{dx} + \frac{y}{x} = x^3$

Sol:

$$p(x) = \frac{1}{x} \Rightarrow \mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$(x)y = \int x^3.x\,dx + c \Rightarrow x\,y = \frac{x^5}{5} + c\,] \quad \div x$$

$$y = \frac{x^4}{5} + \frac{c}{x}$$

Ex: Solve $(1 + x^2)dy + (2x\,y - 4\,x^2)\,dx = 0\,, y(0) = 1$ (P.S.)

Sol:

$$(1 + x^2)\frac{dy}{dx} + 2xy - 4\,x^2 = 0 \qquad \div by\ (1 + x^2)$$

$\frac{dy}{dx} + \frac{2x}{1+x^2}\, y = \frac{-4x^2}{1+x^2}$

$P(x) = \frac{2x}{1+x^2} \Rightarrow \mu(x) = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = 1 + x^2$

$$(1+x^2)y = \int \frac{4x^2}{1+x^2} . (1+x^2)dx + c$$

$$(1+x^2)y = \frac{4}{3}x^3 + c \Rightarrow y = \frac{1}{1+x^2}\left(\frac{4}{3}x^3 + c\right)$$

Substitute the initial condition: $x = 0, y = 1$

$$(1+0).1 = \left(\frac{4}{3}.0 + c\right) \Rightarrow c = 1$$

The complete solution is:

$$y = \frac{1}{1+x^2}\left(\frac{4}{3}x^3 + 1\right)$$

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Ex: Solve the following D.E.

$$\frac{dy}{dx} + 3\, y \cos x = \cos x$$

Sol:

p(x)= 3 cos x , Q(x) =cos x

$$\mu(x) = \int e^{3\cos x\, dx} = e^{3\sin x}$$

$$\mu(x).y = \int e^{3\sin x} \cos x\, dx + c$$

$$e^{3\,\mathrm{si}} .y = \int e^{3\sin x} \cos x\, dx + c \quad \div\ e^{3\sin x}$$

$$y = \frac{1}{e^{3\sin x}}\left[-\frac{1}{3}e^{3\sin x} + c\right]$$

$$= -\frac{1}{3} + \frac{c}{e^{3\sin x}}$$

Sometime the given 1st ODE can be written as:

$$\frac{dx}{dy} + p(y)\,x = Q(y)$$

Then

$$\mu(y) = e^{\int p(y)dy}$$

And the solution is:

$$\mu(y).x = \int Q(y).\mu(y)\,dy + c$$

Ex: Solve $(1+\ y^2) + \left(x\ - e^{tan^{-1}y}\right)\frac{dy}{dx} = 0$

Sol:

$$(1+\ y^2) + \left(x\ - e^{tan^{-1}y}\right)\frac{dy}{dx} = 0 \qquad * \frac{dx}{dy}$$

$$(1+\ y^2)\frac{dx}{dy} + x = e^{tan^{-1}y} \qquad \div\ (1+\ y^2)$$

$$\frac{dx}{dy} + \frac{1}{1+y^2}\,x = \frac{1}{1+y^2}\,e^{tan^{-1}y}$$

$p(y) = \frac{1}{1+y^2}$, $Q(y) = \frac{1}{1+y^2}\,e^{tan^{-1}y}$

$$\Rightarrow \mu(y) = e^{\int \frac{1}{1+y^2}dy} = e^{tan^{-1}y}$$

$$e^{tan^{-1}y}.x = \int \frac{1}{1+y^2}\,e^{tan^{-1}y}.e^{tan^{-1}y}\,dy + c$$

$$e^{tan^{-1}y}.x = \int \frac{1}{1+y^2}\,e^{2tan^{-1}y}dy + c$$

$$e^{tan^{-1}y}.x = \frac{1}{2}\, e^{2tan^{-1}y} + c \qquad \div \;\; e^{tan^{-1}y}$$

$$x = \frac{1}{2}\,\frac{e^{2ta\;\;^{-1}y}}{e^{tan^{-1}y}} + \frac{c}{e^{tan^{-1}y}}$$

$$x = \frac{1}{2}\, e^{tan^{-1}y} + \; c\, e^{-tan^{-1}y}$$

<u>Equation Reduction to Linear equation</u>

1. **Bernoulli's Eq**. (non-linear D.E) which has the following general form:

$$\boldsymbol{\frac{dy}{dx} + p(x).y = Q(x).y^n \ldots\ldots\ldots\ldots..* \qquad n \neq 1 \;, n \neq 0}$$

this eq. has to be transformed to **linear eq**. by changing the dependent variable y according to:

let $z = \; y^{1-n} \Rightarrow \frac{dz}{dx} = (1-n)y^{-n}\,\frac{dy}{dx} \qquad \div\, (1-n)y^{-n}$

$$\frac{dy}{dx} = \frac{1}{1{-}n}\,\frac{1}{y^{-n}}\,\frac{dz}{dx} \qquad\qquad sub.\, in\; eq.(*)$$

$$\frac{1}{1-n}\,y^n\,\frac{dz}{dx} + p(x).y = Q(x)\,y^n \qquad * \quad (1-n)$$

$$\frac{dz}{dx}\,y^n \;+\; (1-n)p(x).y = \;\; y^n\;(1-n)\,Q(x) \qquad \div\; y^n$$

$$\frac{dz}{dx} + \;(1-n)p(x)\,y^{1-n} \;=\; (1-n)\,Q(x)$$

$$\frac{dz}{dx} + \;(1-n)\,p(x)\,.z = \;\;(1-n)\,Q(x) \qquad n \neq 1$$

which is linear equation with dependent var. z

Ex: Solve $(1+x^2)\,y' = xy - y^2$

Sol:

$$(1+x^2)\frac{dy}{dx} - xy = -y^2 \qquad * \frac{1}{1+x^2}$$

$$\frac{dy}{dx} - \frac{x}{1+x^2}\,y = \frac{-1}{1+x^2}y^2$$

$$\therefore p(x) = \frac{-x}{1+x^2} \quad , \quad Q(x) = \frac{-1}{1+x^2}$$

$$n = 2 \;\Rightarrow\; let\ z = y^{1-n}$$

$$\therefore\ z = y^{1-2} = y^{-1} = \frac{1}{y}$$

$$G.F\ is\ \ \frac{dz}{dx} + (1-n)\,p(x)\,.\,z = (1-n)\,Q(x) \qquad n \neq 1$$

$$\frac{dz}{dx} + (1-2)\ \frac{-x}{1+x^2}.\,z = (1-2)\ \frac{-1}{1+x^2}$$

$$\frac{dz}{dx} + \frac{x}{1+x^2}.\,z = \frac{1}{1+x^2}$$ This eq. is linear

$$\therefore \mu(x) = e^{\int \frac{x}{1+x^2}dx} = e^{\frac{1}{2}\ln(1+x^2)} = (1+x^2)^{\frac{1}{2}} = \sqrt{1+x^2}$$

$$\sqrt{1+x^2}\,.\,z = \int \sqrt{1+x^2}.\frac{-1}{1+x^2}\ dx + c$$

$$1+x^2 = \sqrt{1+x^2}\,.\sqrt{1+x^2}$$

$$\sqrt{1+x^2}\,.\,z = \int \frac{-1}{\sqrt{1+x^2}}\ dx + c$$

$$\sqrt{1+x^2}\,.\,z = -sinh^{-1}x + c \qquad * \frac{1}{\sqrt{1+x^2}}$$

$$\frac{1}{y} = \frac{1}{\sqrt{1+x^2}}\ (-sinh^{-1}x + c)$$

- **Special Case of First-Order Linear D.E.**

$$f'(y)\frac{dy}{dx} + p(x)f(y) = Q(x)$$

Let $z = f(y) \Rightarrow \frac{dz}{dx} = f'(y)\frac{dy}{dx}$

Ex: Solve the following D.E $(sec^2 y)\frac{dy}{dx} + 2x\tan y = x^3$

Sol:

Let $z = \tan y \Rightarrow \frac{dz}{dx} = sec^2 y\frac{dy}{dx}$

$$\frac{dz}{dx} + 2x\,.z = x^3 \Rightarrow \quad linear\ eq.$$

$p(x) = 2x \quad Q(x) = x^3$

$$\mu(x) = e^{2\int x\,dx} = e^{x^2}$$

$$\mu(x).z = \int Q(x).\mu(x)\,dx$$

$$\Rightarrow e^{x^2}.z = \int x^3 e^{x^2}\,dx + c$$

$$e^{x^2}.z = \frac{1}{2}\left[x^2 e^{x^2} - e^{x^2}\right] + c \qquad \div e^{x^2}$$

$$z = \frac{1}{2}[x^2 - 1] + \frac{c}{e^{x^2}}$$

$\because Z = \tan y$

$\therefore \quad \tan y = \frac{1}{2}[x^2 - 1] + c\,e^{-x^2}$

2. Exact Equation

A first -order ODE of the form:

$$M\,(x,y)dx + N(x,y)dy = 0$$

Is called **exact** if the following condition is satisfied:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \textit{condition for exactness}$$

Then, the solution is:

$$\int M\,(x,y)dx + \int \textit{only terms free of } x \textit{ in } N(x,y)dy = c$$

EX: Solve $\dfrac{dy}{dx} = -\dfrac{1+e^{\frac{x}{y}}}{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}$

Sol:

$$e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = -\left(1+e^{\frac{x}{y}}\right)dx$$

$$\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$$

$$\left\{\begin{array}{l} M\,(x,y) = 1+e^{\frac{x}{y}} \Rightarrow \dfrac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(-\dfrac{x^2}{y}\right) \\ N(x,y) = e^{\frac{x}{y}}\left(1-\dfrac{x}{y}\right) \Rightarrow \dfrac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(-\dfrac{x}{y^2}\right) \end{array}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The ODE is exact, and the solution is:

$$\int M\,(x,y)dx + \int \textit{only terms free of } x \textit{ in } N(x,y)dy = c$$

$$\int \left(1+e^{\frac{x}{y}}\right)dx = c \quad [\textit{no term in } N\,(x,y) \textit{ is free of } x]$$

$$\therefore x + y\,e^{\frac{x}{y}} = c$$

Ex: Solve $(x+y^2)\,x\,dx+(x^2+y)y\,dy=0$

Sol:

$$\left.\begin{cases} M(x,y)=x^2+xy^2 \Rightarrow \dfrac{\partial M}{\partial y}=2xy \\ N(x,y)=yx^2+y^2 \Rightarrow \dfrac{\partial N}{\partial x}=2xy \end{cases}\right\}\ \textit{the condition is satisfied}$$

The ODE is exact, and the solution is:

$$\int (x^2+xy^2)dx+\int y^2dy=c$$

$$\frac{x^3}{3}+\frac{x^2y^2}{2}+\frac{y^3}{3}=c\ \ or\ 2x^3+3x^2\,y+2y^3=6c$$

Equation Reducible to Exact

If the given ODE is **not an exact** equation this means that:

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

Note: If either N or M in the D.E. contains (sin, cos, tan, tanh, sinh,….., log, ln, e (it may be means that this D.E. is not exact

It is possible to convert the given ODE to an exact equation by multiplying it by a factor called the integrating factor (function of x and/or y).

- If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=g(x)\Rightarrow\mu(y)=e^{\int g(x)dx}$
- If $\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=h(y)\Rightarrow\mu(y)=e^{-\int h(y)dy}$
- If M=y f(xy) and N= x g(xy) ⇨ $\mu(x,y)=\frac{1}{xM-yN}$

Ex: Solve $\left(x\,y^2 - e^{\frac{1}{x^3}}\right)dx - x^2y dy = 0$

Sol:

$$\left\{\begin{array}{l} M = x\,y^2 - e^{\frac{1}{x^3}} \Rightarrow \frac{\partial M}{\partial y} = 2xy \\ N = -\,x^2y \Rightarrow \frac{\partial N}{\partial x} = -2xy \end{array}\right\} \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-x^2y}\left(2xy - (-2xy)\right) = \frac{-4}{x} = g(x)$$

$$\because \mu(x) = e^{\int g(x)dx}$$

$$\therefore \mu(x) = e^{\int \frac{-4}{x}dx} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4} = \frac{1}{x^4}$$

$$general\ eq.* \ \frac{1}{x^4}$$

$$\frac{1}{x^4}\left(x\,y^2 - e^{\frac{1}{x^3}}\right)dx - \frac{1}{x^4}\ x^2y\,dy = 0$$

$$\frac{x\,y^2}{x^4}dx - \frac{1}{x^4}e^{\frac{1}{x^3}}dx - \frac{1}{x^4}\ x^2y\,dy = 0$$

$$\left\{\begin{array}{l} M^* = \frac{y^2}{x^3} - \frac{1}{x^4}\ e^{\frac{1}{x^3}} \Rightarrow \frac{\partial M^*}{\partial y} = \frac{2y}{x^3} \\ N^* = \frac{y}{x^2} \Rightarrow \frac{\partial N^*}{\partial x} = -\frac{-2y}{x^3} = \frac{2y}{x^3} \end{array}\right\} \frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x} \quad exact$$

$$\therefore \int M^*(x,y)dx + \int \left(terms\ free\ of\ x\ in\ N^*(x,y)\right)dy = c$$

$$\int\left(\frac{y^2}{x^3} - \frac{1}{x^4}\ e^{\frac{1}{x^3}}\right)dx = c$$

$$-\frac{y^2x^{-2}}{2} + \frac{1}{3}e^{\frac{1}{x^3}} = c \Rightarrow -\frac{y^2x^{-2}}{2x^2} + \frac{1}{3}e^{\frac{1}{x^3}} = c$$

Ex: Solve $(x\,y^3+y)dx+2(x^2y^2+x+y^4)dy=0$

Sol:

$$\left.\begin{cases} M=x\,y^3+y \Rightarrow \dfrac{\partial M}{\partial y}=3x\,y^2+1 \\ N=2(x^2y^2+x+y^4)\Rightarrow \dfrac{\partial N}{\partial x}=4x\,y^2+2 \end{cases}\right\} \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

$$\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{3x\,y^2+1-4x\,y^2-2}{x\,y^3+y}$$

$$\frac{-(x\,y^2+1)}{y(x\,y^2+1)}=\frac{-1}{y}=h(y)$$

$$\therefore \mu(y)=e^{-\int\frac{-1}{y}dy}=e^{\ln y}=y$$

$$general\ eq.*\ y$$

$$\therefore y\,(x\,y^3+y)dx+2y\,(x^2y^2+x+y^4)dy=0$$

$$(x\,y^4+y^2)dx+(2x^2y^3+2xy+2y^5)dy=0$$

$$\left.\begin{cases} M^*=x\,y^4+y^2 \Rightarrow \dfrac{\partial M^*}{\partial y}=4x\,y^3+2y \\ N^*=2x^2y^3+2x\,y+2y^5\Rightarrow \dfrac{\partial N^*}{\partial x}=4x\,y^3+2y \end{cases}\right\} \frac{\partial M^*}{\partial y}=\frac{\partial N^*}{\partial x}\quad \text{exact}$$

$$\int(x\,y^4+y^2)dx+\int 2y^5dy=c$$

$$\frac{x^2y^4}{2}+xy^2+\frac{1}{3}y^6=c$$

Ex: Solve $(x\,y^2\sin x\,y+y\cos x\,y)dx+(x^2\,y\sin x\,y-x\cos x\,y)dy=0$

Sol:

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

$$M=y(x\,y\sin xy+\cos xy)=yf\,(xy)$$

$$N=x\,(x\,y\sin xy-\cos xy)=x\,g\,(x\,y)$$

$$\mu(x,y) = \frac{1}{x\,M - y\,N}$$

$$= \frac{1}{\mathrm{x\,y(x\,y\sin} xy + \cos xy) - y\mathrm{x}\ (\mathrm{x\,y\sin} x\,y - \cos x\,y)}$$

$$= \frac{1}{x^2y^2 sinx\ y + xy\cos x\ y - x^2y^2 sinx\ y + xy\cos x\ y}$$

$$= \frac{1}{2\ xy\cos xy}$$

$$general\ eq.* \ \frac{1}{2\ xy\cos xy}$$

$$\therefore\ \frac{x\,y^2 sinxy + ycosxy}{2\ xy\cos xy}\ dx + \frac{x^2 ysinxy - xcosx}{2\ xy\cos xy}\ dy = 0$$

$$\frac{x\,y^2 sinxy}{2\ xy\cos xy} dx + \frac{ycosxy}{2\ xy\cos xy} dx + \frac{x^2 ysinxy}{2\ xy\cos xy} dy - \frac{xcosxy}{2\ xy\cos xy}\ dy$$

$$(\frac{y}{2} \tan xy + \frac{1}{2x})\ dx + (\frac{x}{2} \tan xy - \frac{1}{2y})\ dy$$

$$M^* = \frac{y}{2} \tan xy + \frac{1}{2x} \quad ⇨ \frac{\partial M^*}{\partial y} = \frac{1}{2} \tan xy + \frac{xy}{2}\ sec^2\ xy$$

$$\mathrm{N}^* = \frac{\mathrm{x}}{2} \tan \mathrm{xy} - \frac{1}{2\mathrm{y}} \quad ⇨ \frac{\partial N}{\partial x} = \frac{1}{2} \tan xy + \frac{xy}{2}\ sec^2\ xy$$

$$\therefore \frac{\partial \mathrm{M}^*}{\partial \mathrm{y}} = \frac{\partial \mathrm{N}^*}{\partial \mathrm{x}} \quad \text{exact}$$

$$\therefore \int (\frac{\mathrm{y}}{2} \tan \mathrm{xy} + \frac{1}{2\mathrm{x}})\ \mathrm{dx} + \int - \frac{1}{2\mathrm{y}}\ \mathrm{dy} = \mathrm{c}$$

$$-\frac{1}{2} ln\,|\cos xy| + \frac{1}{2}\ ln\ x - \frac{1}{2} ln\ y = c * (-2)$$

$$\ln|\cos \mathrm{xy}| - \ln \mathrm{x} + \ln \mathrm{y} = \ln \mathrm{k} \quad , \ln \mathrm{k} = -2\mathrm{c}$$

$$\ln \frac{\mathrm{y}\cos \mathrm{xy}}{\mathrm{x}} = \ln \mathrm{k} \ ⇨ \ \frac{\mathrm{y}\cos \mathrm{xy}}{\mathrm{x}} = \mathrm{k}$$

Tutorial Sheet

1- Linear Equation

- $(1+\ x^3)y' + 3\,x^2 =\ sin^2 x$
- $x\cos x.\, y' + y(x\sin x + \cos x\ ) = 1$
- $\frac{dy}{dx} + \frac{y}{\sqrt{x(1-x)}} = 1 - \sqrt{x}$
- $2y\,y' + \frac{1}{x}y^2 =\ x^3$

$2 - Bernoulli's\ Equation$

- $y' = 4xy + 16\,x\,y^2\,e^{3x^2}$
- $x^3y^2 + xy)\,dx = dy$
- $y' + y =\ y^{\frac{2}{3}}$
- $Bernoulli's\ Equation\ has\ also\ the\ form$
- $\frac{dx}{dy} + p(y).\,x = Q(y).\,x^n \qquad n \neq 1$

Prove that if we put

$z = x^{1-n}, the\ eq.can\ be\ reduced\ to\ linear\ form:$

$$\frac{dz}{dy} + (1-n)\,p(y).\,z\ = (1-n)Q(y)$$

2- Exact Equation

- $(x^2 +\ y^2)(x\,dx + y\,dy) = (xdy - y\,dx)$
- $y^3\sin 2x\,dx - 3\,y^2\cos 2x\,dy = 0$
- $y\,dx - x\,dy + 3\,x^2y^2e^{x^3}dx = 0$
- $(x\,y^2 + 2\,x^2y^3)dx + (x^2\,y - x^3y^2)dy = 0$