Euler - Cauchy Equation:

It is a type of linear ODE with variable coefficients. It can be solved by changing the independent variable, then it can be transformed into a linear equation with constant coefficient.

Euler – Cauchy eq. has the following form:

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x\, y' + a_0 y = r(x)$$

Let $z = \ln x \;\rightarrow x = e^z \rightarrow \frac{dz}{dx} = \frac{1}{x}$

By chain Rule $\rightarrow \frac{dy}{dx} = \frac{dy}{dz} * \frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} = \frac{1}{x} Dy, \quad D = \frac{d}{dz}$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2y}{dz^2} * \frac{dz}{dx} + \frac{-1}{x^2}\frac{dy}{dz} = \frac{1}{x^2} D(D-1)y$$

$$\frac{d^3y}{dx^3} = \frac{1}{x^3}\; D(D-1)(D-2)y$$

In general, $\rightarrow \frac{d^n y}{dx^n} = \frac{1}{x^n}\; D(D-1)(D-2) \ldots\ldots\ldots.(D-n+1)y$

In the case of 2nd – order eq.: $a_2 x^2 y'' + a_1 x\, y' + a_0 y = r(x)$

It can be transformed into:

$$a_2 \frac{d^2y}{dz^2} + (a_1 - a_2)\frac{dy}{dz} + a_0 y = r(z)$$

Ex: Solve $x^2 y'' - xy' + y = x \ln x$

Sol: $z = ln\, x,\; x = e^z,\; r(z) = z\, e^z$

$a_2 = 1, \qquad a_1 = -1, \qquad a_0 = 1$

$$\frac{d^2y}{dz^2} - 2\frac{dy}{dz} + y = ze^z$$

$\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda_{1,2} = 1 \Rightarrow y_h(z) = C_1 e^z + C_2\, z\, e^z$

$y_p(z) = \; e^z(A_1\, Z + A_0\,)\, Z^2 = \; e^z(A_1\, Z^3 + A_0\, Z^2)$

$y'_p(z) = \; e^z(3\, A_1\, Z^2 + \; 2\, A_0\, Z) + e^z(\, A_1\, Z^3 + \; A_0\; Z^2)$

$y''_p(z) = e^z(6A_1 Z + 2A_0) + e^z(3A_1 Z^2 + 2A_0 Z) + e^z(3A_1 Z^2 +$

$2A_0 Z) + +e^z(A_1 Z^3 + A_0 Z^2)$

$= e^z(6A_1 Z + 2A_0) + 2e^z(3A_1 Z^2 + 2A_0 Z) + e^z(A_1 Z^3 + A_0 Z^2) -$

$2e^z(3A_1 Z^2 + 2A_0 Z) - 2e^z(A_1 Z^3 + A_0 Z^2) + e^z(A_1 Z^3 + A_0 Z^2) = Z e^z$

$$6A_1 = 1 \Rightarrow A_1 = \frac{1}{6} \quad , \quad 2A_0 = 0 \Rightarrow A_0 = 0$$

$y_p(z) = \frac{1}{6} Z^3 e^z$

y(z) $= C_1 e^z + C_2 z e^z + \frac{1}{6} Z^3 e^z$

y(x) $= C_1 x + C_2 \, lnx + \frac{1}{6} x \, (ln\, x)^3$

Ex: Solve $x^2 y'' - xy' - 3y = 2x^3$

Sol: $a_2 = 1$, $a_1 = -1$, $a_0 = -3$, $r(x) = 2x^3$

Let $z = \ln x$, $x = e^z$, $r(z) = 2x^3 \Rightarrow 2(e^z)^3 = 2e^{3z}$

$$\because \; a_2 \frac{d^2y}{dz^2} + (a_1 - a_2)\frac{dy}{dz} + a_0 y = r(z)$$

$$\therefore (1)\frac{d^2y}{dz^2} + (-1-1)\frac{dy}{dz} - 3y = 2e^{3z}$$

$$\frac{d^2y}{dz^2} - 2\frac{dy}{dz} - 3y = 2e^{3z} \dots\dots\dots\dots\dots\dots\dots.(1)$$

$\lambda^2 - 2\lambda - 3 = 0 \Rightarrow (\lambda - 3)(\lambda + 1) = 0$

$\lambda_1 = 3 \quad \lambda_2 = -1$

$y_h(z) = C_1 e^{3z} + C_2 e^{-z}$

$\because \; r(z) = 2e^{3z}$

$$\left\{\begin{array}{c} \therefore y_p(z) = Ae^{3z} \; (z) \Rightarrow A\,z\,e^{3z} \\ y'_p(z) = 3\,A\,z\,e^{3z} + A\,e^{3z} \\ y''_p(z) = 9\,A\,z\,e^{3z} + 3A\,e^{3z} + 3A\,e^{3z} = 9\,A\,z\,e^{3z} + 6A\,e^{3z} \end{array}\right\} inseart\ in\ (1)$$

$$\frac{d^2y}{dz^2} - 2\frac{dy}{dz} - 3y = 2e^{3z} \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

$$9\,A\,z\,e^{3z} + 6A\,e^{3z} - 2(3\;A\,z\,e^{3z} + A\,e^{3z}) - 3\,A\,z\,e^{3z} = 2e^{3z}$$

$$9\,A\,z\,e^{3z} + 6A\,e^{3z} - 6A\,z\,e^{3z} - 2A\,e^{3z} - 3\,A\,z\,e^{3z} = 2e^{3z}$$

$$4A\,e^{3z} = 2e^{3z}$$

$$4A = 2 \Rightarrow A = \frac{1}{2}$$

$$\because y_p(z) = A\,z\,e^{3z}$$

$$\therefore y_p(z) = \frac{1}{2}\,z\,e^{3z}$$

$$\mathrm{y(z)} = y_h + y_p$$

$$\mathrm{y(z)} = C_1 e^{3z} + C_2\;e^{-z} + \frac{1}{2}\;z\,e^{3z}$$

$$\mathrm{y(x)} = C_1 x^3 + C_2 x^{-1} + \frac{1}{2}x^3 \ln x$$

System of Simultaneous Linear ODEs with Constant Coefficients

We shall now consider a system of simultaneous linear differential equations which contains two or more dependent variables such as "x,y,z,…." And a single independent variable such as" t". ". In general, the number of equations equals to the number of dependent variables. Consider the following system of linear ODEs:

$$(a_{11}D^2 + b_{11}D + c_{11})x(t) + (a_{12}D^2 + b_{12}D + c_{12})y(t) = r_1(t)$$

$$(a_{21}D^2 + b_{21}D + c_{21})x(t) + (a_{22}D^2 + b_{22}D + c_{22})y(t) = r_2(t)$$

Where $D = \frac{d}{dt}$, $D^2 = \frac{d^2}{dt^2}$

Re-write the system of ODEs in matrix form:

$|(a_{11}D^2 + b_{11}D + c_{11})\ (a_{12}D^2 + b_{12}D + c_{12})|$

$$\begin{vmatrix} (a_{11}D^2 + b_{11}D + c_{11})\ (a_{12}D^2 + b_{12}D + c_{12}) \\ (a_{21}D^2 + b_{21}D + c_{21})(a_{22}D^2 + b_{22}D + c_{22})y(t) \end{vmatrix} \begin{vmatrix} x(t) \\ y(t) \end{vmatrix} = \begin{vmatrix} r_1(t) \\ r_2(t) \end{vmatrix}$$

$$[f(D)]x(t) = \begin{vmatrix} r_1(t)(a_{12}D^2 + b_{12}D + c_{12}) \\ r_2(t)(a_{22}D^2 + b_{22}D + c_{22}) \end{vmatrix}$$ ←ODE in $x(t)$

$$[f(D)]y(t) = \begin{vmatrix} (a_{11}D^2 + b_{11}D + c_{11})r_1(t) \\ (a_{21}D^2 + b_{21}D + c_{21})r_2(t) \end{vmatrix}$$ ←ODE in $y(t)$

$$f(D) = \begin{vmatrix} (a_{11}D^2 + b_{11}D + c_{11})(a_{12}D^2 + b_{12}D + c_{12}) \\ (a_{21}D^2 + b_{21}D + c_{21})(a_{22}D^2 + b_{22}D + c_{22}) \end{vmatrix}$$ ← (determinants)

<u>Problem 1</u>. Solve the following system of ODEs

$$2\frac{dx}{dt}+\frac{dy}{dt}-4x-y=e^t \quad \Rightarrow (2D-4)x+(D-1)y=e^t$$

$$\frac{dx}{dt}+3x+y=0 \Rightarrow (D+3)x+y=0$$

$$(2D-4)x+(D-1)y=e^t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$(D+3)x+y=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Multiplying **(2)** by (D-1)

$$(2D-4)x+(D-1)y=e^t$$

$$(D-1)(D+3)x+(D-1)y=0$$

and subtracting we get the solution:

$$(2D-4)x-(D-1)(D+3)x=e^t$$

$$(D^2+1)x=-e^t$$

$$\lambda^2+1=0 \quad \Rightarrow \lambda=\pm i$$

$$x_h(t)=c_1\cos t+c_2\sin t$$

$$r(x)=-e^t \qquad x_p(t)=Ae^t$$

$$x'_p(t)=Ae^t \qquad x''_p(t)=A\,e^t \quad \textit{and sub in given ODE}$$

$$(D^2+1)x=-e^t$$

$$(A\,e^t+A\,e^t)=-e^t$$

$$2A\,e^t=-e^t \qquad A=\frac{1}{2}$$

$$x_p(t)=\frac{1}{2}e^t$$

$$\therefore x(t)=x_h(t)+x_p(t)$$

$$x(t)=c_1\cos t+c_2\sin t-\frac{1}{2}e^t$$

To find y: from **eq. 2**

$$(D+3)x+y=0$$

$$y=-(D+3)x$$

$$y=-Dx-3x$$

$$y = -\frac{dx}{dt} - 3x$$

$\because x(t) = c_1 \cos t + c_2 \ \sin t - \frac{1}{2} e^t$

$\therefore \frac{dx}{dt} = \ -c_1 sint + c_2 \cos t - \frac{1}{2} e^t$

$$y = -\left(-c_1 sint + c_2 \cos t - \frac{1}{2} e^t\right) - 3\left(c_1 \cos t + c_2 \ \sin t - \frac{1}{2} e^t\right)$$

$$y(t) = (c_1 - 3c_2) \sin t - (3c_1 + \ c_2) \cos t + 2e^t$$

Thus, the general solution is

$$x(t) = c_1 \cos t + c_2 \ \sin t - \frac{1}{2} e^t$$

$$y(t) = (c_1 - 3c_2) \sin t - (3c_1 + \ c_2) \cos t + 2e^t$$

<u>Problem 2</u>. Solve the following linear system of ODEs:

$$x' - 4x + 3y = t \ldots\ldots\ldots\ldots\ldots\ldots.(\mathbf{1})$$

$$2x + y' + y = e^{-t} \ldots\ldots\ldots\ldots\ldots\ldots.(\mathbf{2})$$

<u>Solution:</u>

Put the equations in to D-operator form

$Dx - 4x + 3y = t \Rightarrow (D-4)x + 3y = t$………….**(3)** $\ast(D+1)$

$2x + Dy + y = e^{-t} \Rightarrow -2x + (D+1)y = e^{-t}$………**(4)** *3

$\therefore \ (D-4)(D+1)x + 3(D+1)y = (D+1)t \Rightarrow$

$$\left.\begin{aligned}(D^2 - 3D - 4)x + 3(D+1)y = 1 + t\\ -6x + 3(D+1)y = 3e^{-t}\end{aligned}\right\} \ by\ subtraction$$

$$(D^2 - 3D - 4)x + 6x = 1 + t - 3e^{-t}$$

$$(D^2 - 3D - 4 + 6)x = 1 + t - 3e^{-t}$$

$$(D^2 - 3D + 2)x = 1 + t - 3e^{-t}$$

$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2$

$$x_h(t) = c_1 e^t + c_2\, e^{2t}$$

$\because \ \mathrm{r(x)} = \ t$

$\therefore x_p(t) = A_1 t + A_0 + Ke^{-t}$

$\therefore x'_p(t) = A_1 - Ke^{-t}$

$\therefore x''_p(t) = -Ke^{-t}$

sub in given ODE

$(D^2 - 3D + 2)x = 1 + t - 3e^{-t}$

$Ke^{-t} - 3(A_1 - Ke^{-t}) + 2(A_1 t + A_0 + Ke^{-t}) = 1 + t - 3e^{-t}$

$Ke^{-t} - 3A_1 + 3Ke^{-t} + 2A_1 t + 2A_0 + 2Ke^{-t} = 1 + t - 3e^{-t}$

$6Ke^{-t} + 2A_1 t + (-3A_1 + 2A_0) = 1 + t - 3e^{-t}$

$6K = -3 \Rightarrow K = -2$

$2A_1 = 1 \Rightarrow A_1 = \frac{1}{2}$

$-3A_1 + 2A_0 = 1 \Rightarrow -3\left(\frac{1}{2}\right) + 2A_0 = 1 \Rightarrow 2A_0 = 1 + \frac{3}{2} \Rightarrow A_0 = \frac{5}{4}$

$\because x_p(t) = A_1 t + A_0 + Ke^{-t}$

$\therefore x_p(t) = \frac{1}{2} t + \frac{5}{4} - 2e^{-t}$

$\therefore x(t) = x_h(t) + x_p(t)$

$$x(t) = c_1 e^t + c_2 e^{2t} + \frac{1}{2} t + \frac{5}{4} - 2e^{-t}$$

From eq. 3

$(D - 4)x + 3y = t$

$3y = t - (D - 4)x$

$$y = \frac{1}{3}\left(t - \frac{dx}{dt} + 4x\right)$$

$\because x(t) = c_1 e^t + c_2 e^{2t} + \frac{1}{2} t + \frac{5}{4} - 2e^{-t}$

$\therefore x'(t) = \frac{dx}{dt} = c_1 e^t + 2c_2 e^{2t} + \frac{1}{2}$

$$y(t) = \frac{1}{3}\left(t - c_1 e^t + 2c_2 e^{2t} + \frac{1}{2} + 4\left(c_1 e^t + c_2 e^{2t} + \frac{1}{2} t + \frac{5}{4} - 2e^{-t}\right)\right)$$

$$y(t) = \frac{1}{3}\left(t - c_1 e^t + 2\,c_2\,e^{2t} + \frac{1}{2} + 4c_1 e^t + 4c_2\,e^{2t} + 2\,t + 5 - 8e^{-t}\right)$$

$$y(t) = \frac{1}{3}\left(3c_1 e^t + 2\,c_2\,e^{2t} + 3\,t + \frac{9}{2} - 8e^{-t}\right)$$

$$y(t) = c_1 e^t + \frac{2}{3}\,c_2\,e^{2t} + t + \frac{3}{2} - \frac{8}{3}e^{-t}$$