2- Euler Beta Function β(x,y):-

Beta Function is a function of two parameters (variables). It is used in statistics applications.

$$\boldsymbol{\beta}(\mathbf{x},\mathbf{y}) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \qquad x, y>0$$

Beta Function can be represented in term of Gamma function: -

$$\boldsymbol{\beta}(\mathbf{x},\mathbf{y}) = \frac{\Gamma(\mathbf{x})\ \Gamma(\mathbf{y})}{\Gamma(\mathbf{x}+\mathbf{y})}$$

- $\beta(x,y) = \beta(y,x)$
- $\beta(x,y) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2x-1}.(\cos\theta)^{2y-1}\,d\theta$
- $\beta(x,y) = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}}\,dt$

EX: prove that: $\int_0^{\frac{\pi}{2}} (\sin\theta)^{2x-1}.(\cos\theta)^{2y-1}\,d\theta = \frac{1}{2}\beta(x,y)$

Sol:

General form is $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$

Let $t = sin\theta^2 \Rightarrow \sin\theta = \sqrt{t} \Rightarrow \quad \therefore 1-t = 1-sin\theta^2 = cos\theta^2$

$dt = 2\sin\theta\, cos\theta\, d\theta \qquad\qquad \cos\theta = \sqrt{1-t}$

$\cos\theta\, d\theta = \frac{1}{2}\frac{1}{\sin\theta}\,dt$

$\cos\theta\, d\theta = \frac{1}{2}\frac{1}{\sqrt{t}}\,dt$

$\cos\theta\, d\theta = \frac{1}{2}\, t^{-\frac{1}{2}}\ dt$

$d\theta = \frac{1}{2}\frac{t^{-\frac{1}{2}}}{\cos\theta}\ dt$

$\therefore d\theta = \frac{1}{2}\frac{t^{-\frac{1}{2}}}{\sqrt{1-t}}\ dt$

$\int_0^{\frac{\pi}{2}} (\sin\theta)^{2x-1} . (\cos\theta)^{2y-1}\, d\theta$

We have $\sin\theta = \sqrt{t}$

$\cos\theta = \sqrt{1-t}$

Therefore,

$\int_0^1 (\sqrt{t})^{2x-1}\, (\sqrt{1-t})^{2y-1} .\ \frac{1}{2} . \frac{t^{-\frac{1}{2}}}{\sqrt{1-t}}\ dt$

$I = \int_0^1\ (t)^{\frac{2x-1}{2}}\ . (1-t)^{\frac{2y-1}{2}} . \frac{1}{2}\ \frac{t^{-\frac{1}{2}}}{\sqrt{1-t}}\ dt$

$= \frac{1}{2}\ \int_0^1\ t^{x-1} . (1-t)^{y-1}\, dt = \frac{1}{2}\beta(x, y)$

EX: prove that: $\int_0^{\infty}\ \frac{t^{x-1}}{(1+t)^{x+y}}\ dt\ = \frac{1}{2}\beta(x, y)$

Sol:

Let $z = \frac{t}{1+t}$ ⇨ $dt = \frac{dz}{(1-z)^2}\begin{cases} t = 0 & ⇨\ z = 0 \\ t = \infty & ⇨\ z = 1 \end{cases}$

$\therefore I = \int_0^1\ (\frac{z}{1-z})^{x-1} . \frac{1}{(1+\frac{z}{1-z})^{x+y}}\ . \frac{1}{(1-z)^2}\ dz$

$\int_0^1\ (\frac{z}{1-z})^{x-1} . \frac{1}{(\frac{1-z+z}{1-z})^{x+y}} . \frac{1}{(1-z)^2}\ dz$

$$\int_0^1 (z)^{x-1}\,(1-z)^{-(x-1)}\,(1-z)^{x+y}\,(1-z)^{-2}dz$$

$$I = \int_0^1 (z)^{x-1}\,(1-z)^{y-1}\,dz$$

$$I = \beta(x, y) = \frac{\Gamma(x)\;\Gamma(y)}{\Gamma(x+y)}$$

$$\therefore \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt = \frac{1}{2}\beta(x, y)$$

EX: Evaluate: $I = \int_0^2 x.\ \sqrt[3]{8-x^3}\ dx$

Sol:

Let $x^3 = 8t \Rightarrow x = \sqrt[3]{8}\, t^{\frac{1}{3}} \Rightarrow dx = \frac{2}{3}\, t^{\frac{-2}{3}}\, dt$

$$I = \int_0^1 2\,t^{\frac{1}{3}}\ \sqrt[3]{8\,(1-t)}\,.\frac{2}{3}\ t^{\frac{-2}{3}}\ dt$$

$$I = \int_0^1 2\,t^{\frac{1}{3}}\ \sqrt[3]{8}\ (1-t)^{\frac{1}{3}}\,.\frac{2}{3}\ t^{\frac{-2}{3}}\ dt$$

$$\frac{8}{3}\int_0^1 t^{-\frac{1}{3}}\,(1-t)^{\frac{1}{3}}\ dt$$

$$\int_0^1 t^{x-1}.\,(1-t)^{y-1}\,dt$$

From general integral

$\because x - 1 = -\frac{1}{3} \qquad \therefore x = \frac{2}{3}$

$\because y - 1 = \frac{1}{3} \qquad \therefore y = \frac{4}{3}$

$$\therefore\ \frac{8}{3}\,\beta\left(\frac{2}{3}, \frac{4}{3}\right) = \frac{8}{3}\,\frac{\Gamma\left(\frac{2}{3}\right)\ \Gamma\left(\frac{4}{3}\right)}{\Gamma(2)}$$

$\because \Gamma(2) = 1$

$$= \frac{8}{3} \frac{\Gamma\left(\frac{2}{3}\right) \; \Gamma\left(\frac{4}{3}\right)}{1}$$

EX: Evaluate: $I = \int_0^{\frac{\pi}{2}} (tanx)^{3.2} \; dx$

Sol:

$$I = \int_0^{\frac{\pi}{2}} \frac{(sinx)^{3.2}}{(cosx)^{3.2}} \; dx$$

$$= \int_0^{\frac{\pi}{2}} (sin)^{3.2} . (cos)^{-3.2} \; dx$$

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{2x-1} . (\cos\theta)^{2y-1} \, d\theta = \frac{1}{2} \beta\,(x, y)$$

From general integral of trigonometric function

$2x - 1 = 3.2 \;\; \Rightarrow x = 2.1$

$2y - 1 = -3.2 \Rightarrow y = -1.1$

$\mathrm{x} + \mathrm{y} = 2.1 + (-1.1) = 1$

$$= \frac{1}{2} \beta\,(2.1, -1.1) = \frac{1}{2} . \frac{\Gamma(2.1)\Gamma(-1.1)}{1}$$

Evaluate: $I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta} \; \mathrm{d}\theta$

Sol:

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{2x-1} . (\cos\theta)^{2y-1}\, d\theta = \frac{1}{2} \beta\,(x, y)$$

$$I = \int_0^{\frac{\pi}{2}} (sin\theta)^{\frac{1}{2}} \; (\cos\theta)^0 \; d\theta$$

$$2x - 1 = \frac{1}{2} \quad \Rightarrow x = \frac{3}{4}$$

$$2y - 1 = 0 \Rightarrow y = \frac{1}{2}$$

$$\mathrm{x} + \mathrm{y} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

$$\therefore \; \frac{1}{2} \beta \left(\frac{3}{4}, \frac{1}{2}\right)$$

$$\frac{1}{2} . \frac{\Gamma\left(\frac{3}{4}\right) \; \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} = \frac{1}{2} \; \frac{\frac{4}{3} \Gamma\left(\frac{7}{4}\right) \sqrt{\pi}}{\Gamma\left(\frac{5}{4}\right)}$$

$\Gamma\left(\frac{7}{4}\right)$, $\quad \Gamma\left(\frac{5}{4}\right)$ *are from table*

Special Function Tutorial sheet

- Gamma Function

1. Find $\Gamma(5.2)$, $\Gamma(-4.6)$, $\Gamma(4.5)$, $\Gamma(-4.5)$

   Answer: 32.5781 , −0.0537 , 11.6317 , −0.06

2. Evaluate the following Integrals:-

   a. $\int_0^\infty (x+1)^2 e^{-x^2} dx$ ans: $\frac{3}{4}\sqrt{\pi}+1$

   c. $\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{z-1} dx$ ans: $\Gamma(z)$

   b. $\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$ ans: 2

   d. $\int_0^\infty e^{-x^3} dx$ ans: $\Gamma(\frac{4}{3})$

- Beta Function: Evaluate the following integrals:

a. $\int_0^\infty \frac{x^{a-1}}{1+x^4} dx$ ans: $\frac{1}{4} \frac{\pi}{\sin(\frac{\pi a}{4})}$

b. $\int_0^1 (1-z^k)^{1/k} dz$ ans: $\frac{1}{k} \frac{[\Gamma(\frac{1}{k})]^2}{2\Gamma(\frac{2}{k})}$

c. $\int_0^\infty \frac{\sqrt{z}}{(2+bz^2)^2} dz$ ans: $\frac{\pi}{2^{15/4} \cdot b^{3/4}}$