Response of Single Degree of Freedom System to Harmonic Excitation

Frequency domain techniques are more suitable for treating the response to periodic excitation than time domain techniques.

The equation of motion of a damped single degree of freedom system shown in Figure (1)

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \qquad \cdots\cdots\cdots (1)$$

where

$F(t)$: – applied harmonic force (harmonic excitation)

Equation (1) is non-homogenous second order D.E, then the general solution of this equation is

$$x(t) = x_h + x_p$$

where x_h: – homogenous (free vibration)

x_p: – particular solution (forced vibration)

let $F(t) = kf(t) = kA\cos\omega t \qquad \cdots\cdots\cdots (2)$

where $f(t) = A\cos\omega t \ \ (m)$

or $F(t) = F_0 \cos\omega t$

in which ω: – excitation frequency, or driving frequency

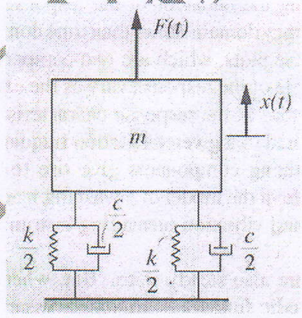


Figure ()

Response of Un-damped System to Harmonic Excitation

If the system is undamped $(c = 0)$ the Equation (1) becomes to

$$m\ddot{x}(t) + kx(t) = F(t) \qquad \cdots\cdots\cdots (1)$$

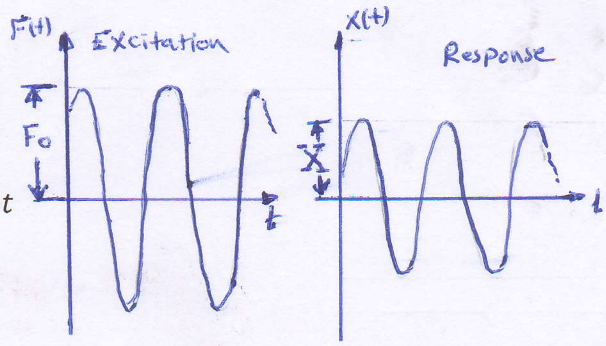
where $x_h = C_1 \cos\omega_n t + C_2 \sin\omega_n t \quad , \quad \omega_n = \sqrt{\frac{k}{m}}$

because the excitation force $F(t)$ is harmonic with frequency ω, the particular solution x_p is also harmonic and has same frequency ω, let

$$x_p = X \cos\omega t$$

where, X: – is amplitude of steady-state response

$\dot{x}_p = -X\omega \sin\omega t \qquad \ddot{x}_p = -X\omega^2 \cos\omega t$



then, from Equation (1)

$$-mX\omega^2 \cos\omega t + kX \cos\omega t = F_0 \cos\omega t$$

$X = \frac{F_0}{k - m\omega^2} \qquad$ then $\qquad x_p = \frac{F_0}{k - m\omega^2} \cos\omega t$

so $$x(t) = \underbrace{C_1 \cos \omega_n t + C_2 \sin \omega_n t}_{x_h} + \underbrace{\frac{F_0}{k - m\omega^2} \cos \omega t}_{x_p}$$

where C_1 and C_2 are constants which are evaluated from initial conditions $x(t=0) = x_0$, $\dot{x}(t=0) = v_0$

so $$C_1 = x_0 - \frac{F_0}{k - m\omega^2} \qquad C_2 = \frac{v_0}{\omega_n}$$

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right) \cos \omega_n t + \left(\frac{v_0}{\omega_n}\right) \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t \quad \ldots\ldots\ldots(2)$$

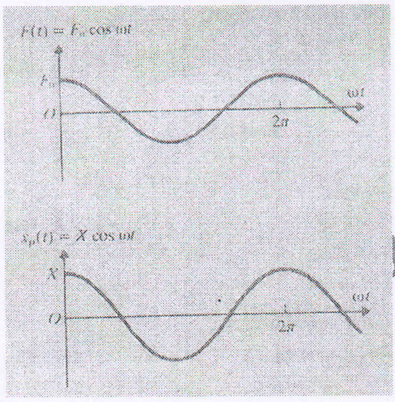
but $$X = \frac{F_0}{k - m\omega^2} \times \frac{k}{k} \qquad \frac{Xk}{F_0} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} = \textbf{Amplitude Ratio}$$

Notes

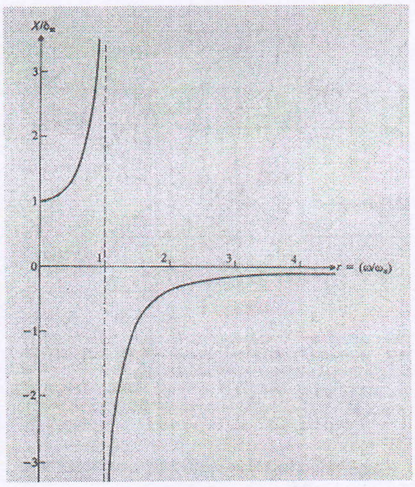
1- when $\frac{\omega}{\omega_n} < 1$ the amplitude ratio is positive and is in phase with external excitation

2- when $\frac{\omega}{\omega_n} > 1$ the amplitude ratio is 180° out of phase with external excitation

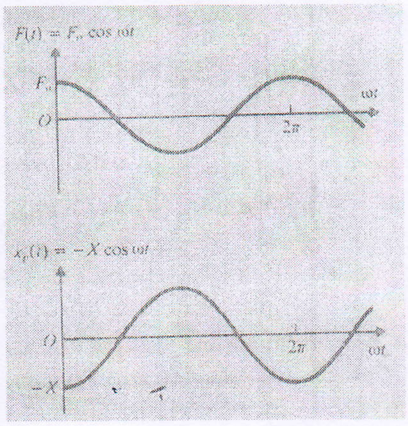
3- when $\frac{\omega}{\omega_n} = 1$ the amplitude ratio → ∞ called **resonance**



$\frac{\omega}{\omega_n} < 1$



$\frac{\omega}{\omega_n} = 1$



$\frac{\omega}{\omega_n} > 1$

Response of a Damped System to Harmonic Excitation

If the system is damped ($c \neq 0$) the Equation (1) becomes to

The equation of motion of a damped single degree of freedom system shown in Figure (1)

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \qquad \cdots\cdots\cdots (1)$$

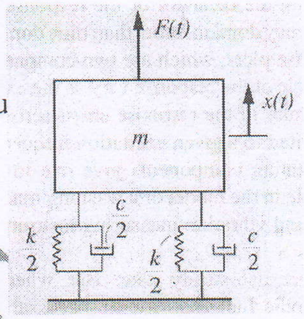


Figure ()

Equation (1) is non-homogenous second order D.E, then the general solu

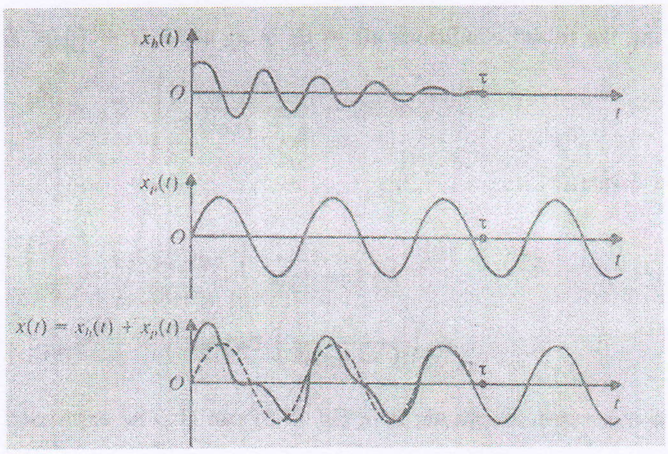
$$x(t) = x_h + x_p$$

where x_h: − homogenous (free vibration)

x_p: − particular solution (forced vibration)

let $F(t)$ be a harmonic $F(t) = F_0 \cos \omega t$

in which ω: − excitation frequency, or driving frequency



From the figure the displacement of mass $(m) = x$

Applying Newton's second law

$$-kx - c\dot{x} + F_0 \cos \omega t = m\ddot{x}$$

After re arranging $m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t \qquad \cdots\cdots\cdots (1)$

From homogeneous part $\omega_n^2 = \frac{k}{m}$ and $\frac{c}{m} = 2\zeta\omega_n$

Now let $x(t) = X \cos(\omega t - \phi)$ = steady-state response ϕ:- is phase angle between response and excitation

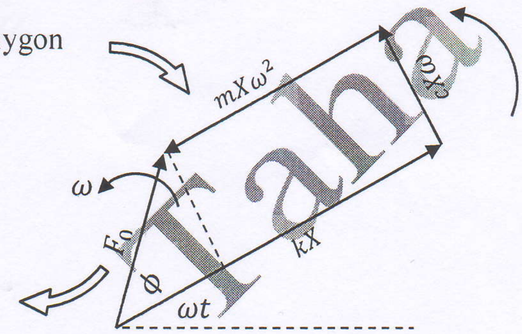
substituting (t) , $\dot{x}(t)$ and $\ddot{x}(t)$ into equation (1)

$$-mX\omega^2\cos(\omega t-\phi)-cX\omega\sin(\omega t-\phi)+kX\cos(\omega t-\phi)=F_0\cos\omega t$$

or $$-\underbrace{mX\omega^2}_{\substack{inertia\\force}}\cos(\omega t-\phi)-\underbrace{cX\omega}_{\substack{dacemping\\for}}\cos\left(\omega t-\phi+\frac{\pi}{2}\right)+\underbrace{kX}_{\substack{spring\\force}}\cos(\omega t-\phi)=F_0\cos\omega t$$

$$(kX-mX\omega^2)\cos(\omega t-\phi)-cX\omega\cos\left(\omega t-\phi+\frac{\pi}{2}\right)=F_0\cos\omega t$$

From this equation we can graph the following force polygon



then from this polygon we can extract that

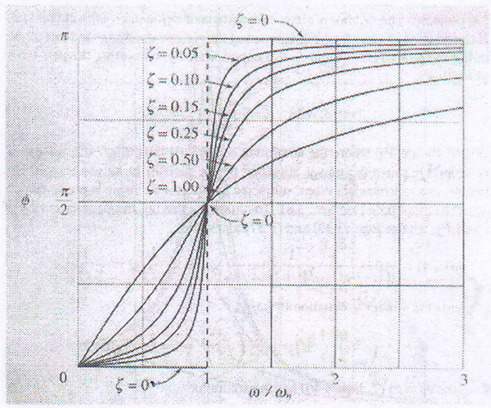
$$X=\frac{F_0}{\sqrt{[k-m\omega^2]^2+[c\omega]^2}}\times\frac{k}{k}$$

since $\omega_n^2=\frac{k}{m}$, $\frac{c}{m}=2\zeta\omega_n$ and $\frac{c\omega}{k}=2\zeta\frac{\omega}{\omega_n}$

so $$\boxed{\frac{Xk}{F_0}=\frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}}=\textbf{amplitude ratio}}\quad\cdots\cdots\cdots(2)$$

and $$\tan\phi=\frac{c\omega}{k-m\omega^2}\times\frac{k}{k}=\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

$$\boxed{\phi=\tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}}$$



to sketch the amplitude ratio $\frac{Xk}{F_0}$ with $\frac{\omega}{\omega_n}$

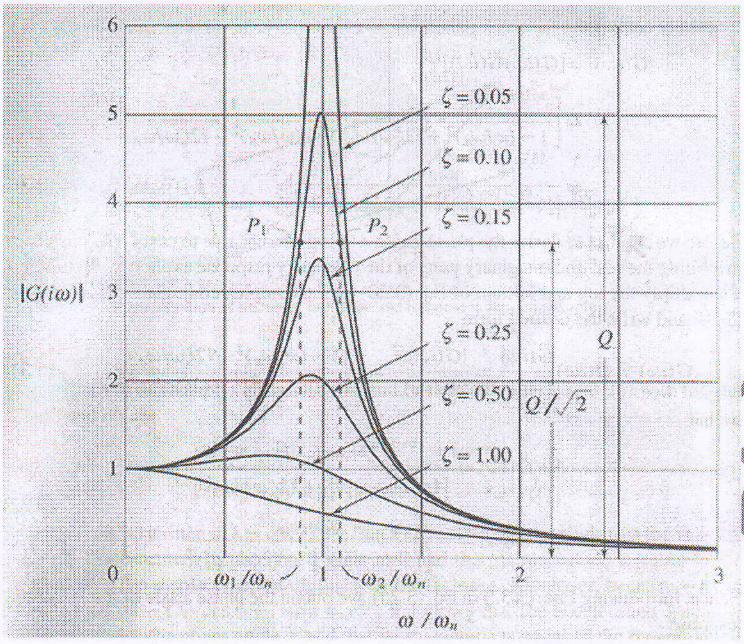
let $\frac{\omega}{\omega_n}=0$ $\frac{Xk}{F_0}=1$ for any value of ζ

and $\frac{\omega}{\omega_n}=1$ $\frac{Xk}{F_0}=\frac{1}{2\zeta}$ for any value of ζ

to find the maximum value $$\frac{d\left(\frac{Xk}{F_0}\right)}{d(\omega/\omega_n)}=0=-\frac{1}{2}\frac{2[1-(\omega/\omega_n)^2](-2\omega/\omega_n)+2(2\zeta\omega/\omega_n)2\zeta}{\{[1-(\omega/\omega_n)^2]^2+(2\zeta\omega/\omega_n)\}^{3/2}}$$

this yield $\frac{\omega}{\omega_n}=\sqrt{1-2\zeta^2}$ to the left of $\frac{\omega}{\omega_n}=1$

$\frac{\omega}{\omega_n}\to\infty$ $\frac{Xk}{F_0}\to 0$ for any value of ζ



To find the maximum value of amplitude ratio substitute $\frac{\omega}{\omega_n} = \sqrt{1-2\zeta^2}$ in Equation (2) we obtain

$$\left.\frac{Xk}{F_0}\right|_{max} = \frac{1}{2\zeta\sqrt{1-2\zeta^2}} \quad \cdots\cdots\cdots (3)$$

Then the steady state response $x(t) = \frac{F}{k_0}\frac{\cos(\omega t-\phi)}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$

Example

An electric motor of mass 68 kg is mounted on an isolator block of mass 1200 kg and the natural frequency of the total assembly is 160 cpm with a damping factor of $\zeta = 0.1$. If there is an unbalance in the motor that results in a harmonic force of $F = 100\sin 31.4t$, determine the amplitude of vibration of the block and the force transmitted to the floor.

Solution

From Equation (2)

$$\frac{Xk}{F_0} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}} = \textbf{amplitude ratio} \quad \Rightarrow \quad X = \frac{F_0/k}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$$

from given data $\omega = 31.4\ rad/sec$ $F_0 = 100\ N$ $m = 1200 + 68 = 1268$ kg

$\omega_n = 160\text{ cpm} \times \frac{2\pi}{60} = 16.75\ rad/sec$ so $\frac{\omega}{\omega_n} = \frac{31.4}{16.75} = 1.8746$

then $k = \omega_n^2 \times m = (16.75)^2 \times 1268 = 355753\ N/m$