TIKRIT University

Msc. Advance Vibrations Course

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References:

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3. ***Fundamentals of vibrations, Leonard Meirovtich, 2001.***
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1.Multi-Degree of freedom systems:

1.1Introduction:

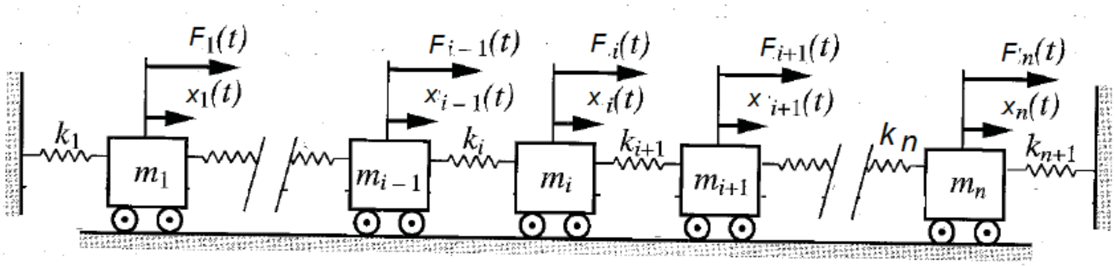
If we defined the degree of freedom by n, which means that if a system has $n \geq 3$, then it will called a multi-degree of freedom system.

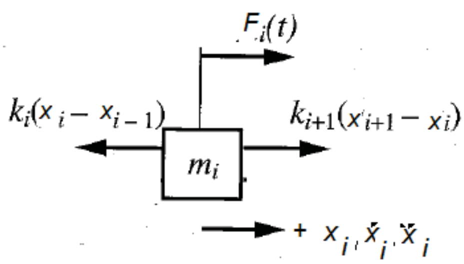
In this case and even in single degree of freedom , and two degree of freedom , each degree of freedom leads to an equation of motion and a natural frequency. The equation of motion of the system can be obtained by many methods, let us take for example:

a) Newton 2^{nd} law of motion.
b) The influence coefficients.
c) Lagrange's equation.

1.2 Newton 2^{nd} law of motion:

Let us consider a simple linear undamped n degree of freedom system as shown below:





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The general equation of motion can be expressed by using Newton 2^{nd} law.

$$\sum F_i = m_i \ddot{x}_i$$

$m_i\ddot{x}_i$ = - $k_i(x_i$ - $x_{i-1}) + \; k_{i+1}(x_{i+1} - x_i) + \; F_i$, i = 1,2,3,…,n-1

Or

$m_i\ddot{x}_i$ - $k_i x_{i-1}$ + (k_i + k_{i+1}) x_i - $k_{i+1}x_{i+1}$ = F_i ….. (1)

And the equations of motion for all masses from m_1 through m_n can be derived by setting i=1 , X_0= 0 , and i = n , X_{n+1}= 0 in equation (1)

Hence

$m_1\ddot{x}_1$ +(k_1+k_2) $X_1 - k_2X_2 = F_1$ ….(2)

$m_n\ddot{x}_n$ - $k_n x_{n-1}$ + (k_n + k_{n+1}) $x_n = F_n$ …… (n)

Equations (2) to (n) can be expressed in a matrix form as:

$[m]\{\ddot{x}\}$ + $[k]\{x\} = \{F\}$

Where

$[m]$ = mass matrix

$[k]$ = stiffness matrix

$\{\ddot{x}\}$ = acceleration vector

$\{x\}$ =displacement vector

$\{F\}$ = general force vector

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_n \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \vdots \\ \ddot{x}_n \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 & 0 & 0 \\ -k_2 & (k_2+k_3) & -k_3 & 0 \\ 0 & -k_3 & (k_3+k_4) & 0 \\ 0 & 0 & -k_n & (k_n+k_{n+1}) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_n \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{Bmatrix}$$

In most general matrix the mass and stiffness matrices are:

$$[m] = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1n} \\ m_{12} & m_{22} & m_{23} & \cdots & m_{2n} \\ \vdots & & & & \\ m_{1n} & m_{2n} & m_{3n} & \cdots & m_{nn} \end{bmatrix}$$

and

$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{12} & k_{22} & k_{23} & \cdots & k_{2n} \\ \vdots & & & & \\ k_{1n} & k_{2n} & k_{3n} & \cdots & k_{nn} \end{bmatrix}$$

1.3 **<u>The influence coefficients</u>:**

Let the system shown in fig(2) be acted on by one force (F_j), and the displacement at point i (m_i) due to F_j is X_{ij} , the

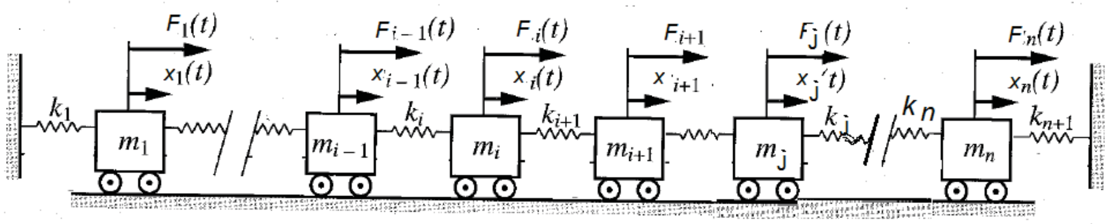


Fig. (2)

Influence coefficients have two type.

One is called flexibility influence coefficient(a_{ij}) , and the other called stiffness influence coefficient(k_{ij}).

For the first one (a_{ij}) is the deflection (displacement) at point i due to a unit load at point j, and can be expressed as:

$X_{ij} = a_{ij}F_j$

For several force (i.e j= 1,2,3,..n) act at deferent points of the system, the total displacement at ANY point can be expressed as:

$X_i = \sum_{j=1}^{n} X_{ij} = \sum_{j=1}^{n} a_{ij}\, F_j$, i =1,2,3,…,n

And in a matrix form

$\{X\} = [a]\{F\}$ …… (1)

Where $[a]$ is the flexibility matrix can be shown as:

$$[a] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

And for the stiffness coefficient (k_{ij}) could be the same case because it can be defined as the force at point i due to a unit displacement at point j, and the total force F_i can be expressed as:

$F_i = \sum_{j=1}^{n} k_{ij} X_j$, i=1,2,3,…n

And in matrix form:

$\{F\} = [K]\{X\}$ …………. (2)

where

$$[k] = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & & & \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix}$$

Subs. Eqn. (2) in (1)

$\{X\} = [a]\{F\} = [a][K]\{X\}$

It mean that

$[a][K] = [I]$ …………(3)

Where

$[I]$ is a unity matrix , eqn. (3) is equivalent to:

$[k] = [a]^{-1}$, $[a] = [k]^{-1}$

<u>Ex :1.1</u>

A three –degree of freedom system shown in fig (3) below, calculate the flexibility and stiffness matrix.

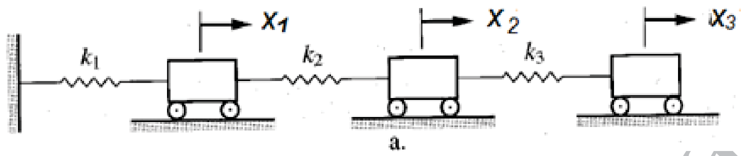
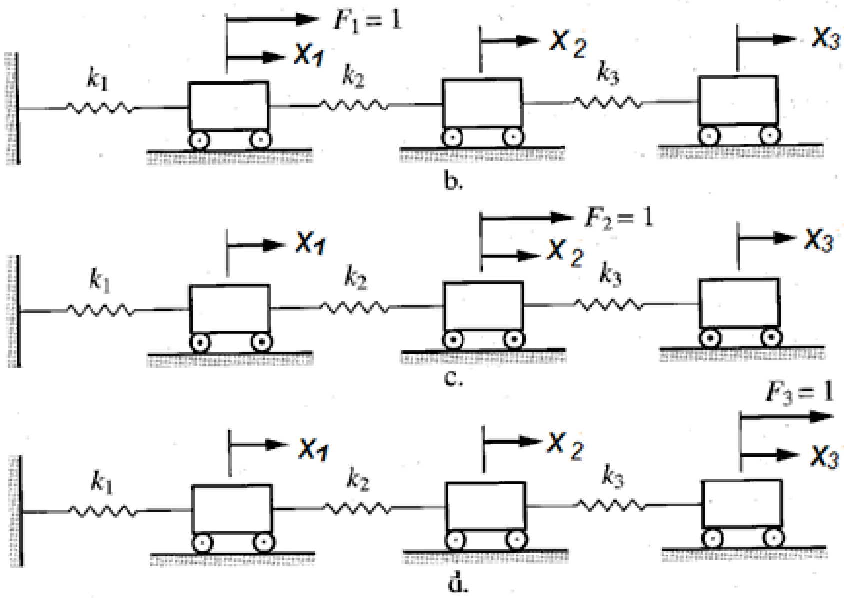


Fig. (3)

Solution:

When we apply F_1= 1, F_2= 0 , F_3= 0



$a_{11} = x_1 = \frac{1}{k_1}$, $a_{21} = x_2 = x_1 = \frac{1}{k_1}$, $a_{31} = x_3 = x_1 = \frac{1}{k_1}$

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{k_1} & 0 & 0 \\ \frac{1}{k_1} & 0 & 0 \\ \frac{1}{k_1} & 0 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

and when $F_2 = 1$, $F_1 = 0$, $F_3 = 0$

$a_{12} = x_1 = \frac{1}{k_1}$, $a_{22} = x_2 = \frac{1}{k_1} + \frac{1}{k_2}$, $a_{32} = x_3 = \frac{1}{k_1} + \frac{1}{k_2}$

or

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} 0 & \frac{1}{k_1} & 0 \\ 0 & \left(\frac{1}{k_1} + \frac{1}{k_2}\right) & 0 \\ 0 & \left(\frac{1}{k_1} + \frac{1}{k_2}\right) & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

And when $F_3=1$, $F_1=0$, $F_2=0$

$a_{31} = x_3 = \frac{1}{k_1}$, $a_{23} = x_2 = \frac{1}{k_1} + \frac{1}{k_2}$, and , $a_{33} = x_2 = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$

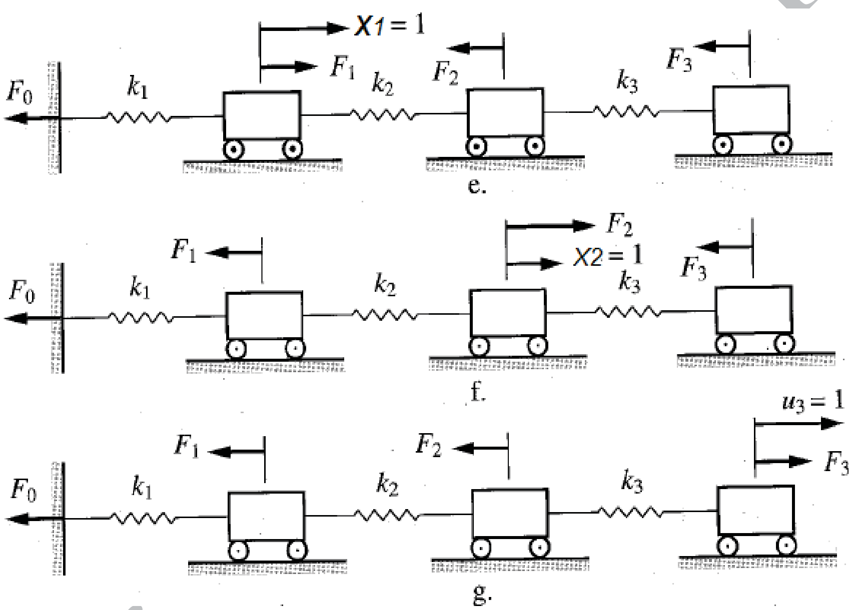
$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{k_1} \\ 0 & 0 & \left(\frac{1}{k_1} + \frac{1}{k_2}\right) \\ 0 & 0 & \left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}\right) \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

To find the final it can be obtained by summation of the above matrices as follows:

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{k_1} & \frac{1}{k_1} & \frac{1}{k_1} \\ \frac{1}{k_1} & \left(\frac{1}{k_1}+\frac{1}{k_2}\right) & \left(\frac{1}{k_1}+\frac{1}{k_2}\right) \\ \frac{1}{k_1} & \left(\frac{1}{k_1}+\frac{1}{k_2}\right) & \left(\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}\right) \end{bmatrix}}_{[a]} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

For stiffness matrix:

$X_1 = 1$, $X_2 = X_3 = 0$



So

$k_{11} = F_1 = k_1 + k_2$, $k_{21} = F_2 = -k_2$, $k_{31} = F_3 = 0$

$$\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \begin{bmatrix} k_1 + k_2 & 0 & 0 \\ -k_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

And $X_2 = 1$, $X_1 = X_3 = 0$

$K_{12} = F_1 = -k_2$, $k_{22} = F_2 = k_2 + k_3$, $K_{32} = F_3 = -k_3$

$$\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \begin{bmatrix} 0 & -k_2 & 0 \\ 0 & k_2 + k_3 & 0 \\ 0 & -k_3 & 0 \end{bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

And $X_3=1$, $X_1= X_2 = 0$

$$\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -k_3 \\ 0 & 0 & k_3 \end{bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

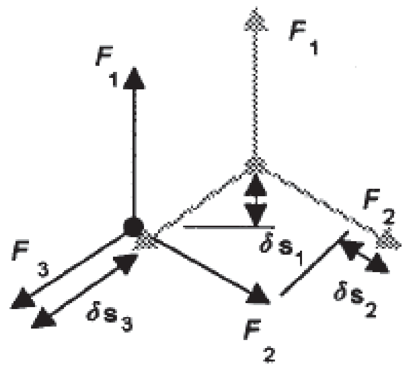
Finally the stiffness matrix is obtained by summation of the above matrices as follows:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

2.Principal of virtual work:

2.1Virtual work.

Virtual work on a system is the work resulting from either virtual forces acting through a real displacement or real forces acting through a virtual displacement, the term displacement may refer to a translation or rotation, and the term force to a force or moment.
The principle of virtual work is if a system of forces acts on a particle which is in static equilibrium and the particle is given any virtual displacement consistent with the constraints imposed by the system then the net work done by the force is zero.
So if a body in equilibrium and subjected to a numbers of forces such as F_1,F_2, and F_3 as shown in the figure below , then it is given an arbitrary displacement $\delta s_1, \delta s_2, \delta s_3$ in the direction of these forces.



Then the virtual work principle gives:

$$F_1\ \delta s_1 + F_2\ \delta s_2 + F_3\ \delta s_3 = 0$$

If the resultant of the above forces is F then with a resultant displacement δs in the direction of the resultant force, the virtual work principle would mean that:

$F\,\delta s = 0$

This can only be the case if either F or δs is 0. Since δs need not be zero then we must have zero resultant force F and so the condition for static equilibrium.

In applying the principle of virtual work to a system of interconnected rigid bodies:

- Externally applied loads are forces capable of doing virtual work when subject to a virtual displacement.
- Reactive forces at fixed supports will do no virtual work since the constraints of the system mean that they are not capable of having virtual displacements.
- Internal forces in members always act in equal and oppositely directed pairs and so a virtual displacement will result in the work of one force cancelling out the work done by the other force. Thus the net work done by internal forces is zero.

2.2 Generalized Coordinate & Forces:

In more complex systems it is of an convenient to describe the system in terms of coordinates some of which may not be independent. Such coordinates may be related to each other by constraint equations . The generalized coordinates may be length, angle, or any other set of numbers that define uniquely the dynamic configuration of the system. Usually designated by $q_1, q_2, \ldots q_n$.

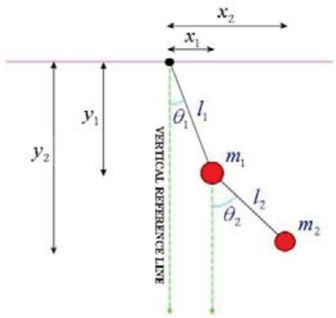
The Generalized forces in general certain forces act on the system when the Generalized Coordinates q_j is changed by small a mount δq_j , the work done can be denoted as U_j . Then the generalized force, Q_j corresponding to q_j can be defined as:

$$Q_j = \frac{U_j}{\delta q_j} \qquad j = 1,2, \ldots \ldots n$$

Note that if q_j is linear displacement then Q_j is a force , and when it is an angular displacement then Q_j is a moment.

Example:

Double pendulum



1. θ_1, and θ_2 are generalized coordinate , then $\theta_1 = q_1$, $\theta_2 = q_2$
2. The position of m_1, m_2 can also be expressed in rectangular x ,y they related by constraints .
   $l_1^2 = x_1^2 + y_1^2$, $l_2^2 = (x_2 - x_1)^2 + = (y_2 - y_1)^2$
3. The rectangular coordinates x ,y can also expressed in term of generalized coordinate θ_1, θ_2 or
   $x_1 = l_1 \sin\theta_1$, $x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$
   $y_1 = l_1 \cos\theta_1$, $y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$

   Number of degree of freedom (n)= Number of dependent coordinates – Number of constrain equation.

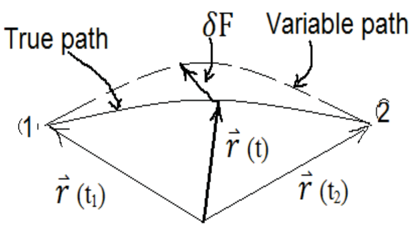
   So for (2 above) , Number of dependent coordinates is 4 (x_1, x_2, y_1, y_2), and 2 constrain equations.
   n= 4-2= 2 DOF
   and for (3 above) , Number of dependent coordinates is 6 (x_1, x_2, y_1, y_2, θ_1, θ_2), and 4 constrain equations
   n= 6-4= 2 DOF

2.3Hamilton's principal.

Consider a mass particle moving in a force field $\vec{F}$ and let $\vec{r}$ (t) is the instantaneous position vector of the particle. From Newton 2nd law in vector form , the actual path of the particle is described by the equation.

$m\vec{a}$ =$\vec{F}$ or $m\frac{d^2r}{dt^2} - F = 0 \ldots\ldots\ldots\ldots\ldots .(1)$

Now, consider any other paths where the particle is located at t = t_1 and



t = t_2 , such a path is of course described by the vector function
$\delta\vec{r}|_{t1} = \delta\vec{r}|_{t2} = 0$

Now, the scalar product of $\delta\vec{r}$ and the term of equation (1) can be formed as

$\{m\ddot{\vec{r}} \,.\, \delta\vec{r}$ - $\vec{F}.\,\delta\vec{r}\} = 0$

Integrating from t_1 to t_2 yield:

$\int_{t_1}^{t_2} m\ddot{\vec{r}}\ \ \delta\vec{r}\,dt - \int_{t_1}^{t_2} \vec{F}.\delta\vec{r}\,dt = 0 \ldots\ldots\ldots\ldots\ldots (2)$

Appling integration by parts to the first term of equation (2)

Let u= $\delta\vec{r}$ and $dv = \ddot{r}\,dt \ \Rightarrow v = \dot{r}$

$$\frac{du}{dt} = \frac{d}{dt}\delta\vec{r} = \delta\frac{d\vec{r}}{dt} = \delta\dot{r} \quad \Rightarrow \quad du = \delta\dot{r}dt$$

$$\therefore m\int_{t_1}^{t_2} \ddot{\vec{r}}\ \ \delta\vec{r}\,dt = \cancelto{0}{m\dot{r}\delta\dot{r}\Big|_{t_1}^{t_2}} - \ m\int_{t_1}^{t_2} \dot{r}\delta\dot{r}dt$$

But $m\dot{r}.\ \delta\dot{r} = \frac{m}{2}\ \delta(\vec{\dot{r}}\,.\vec{\dot{r}}) = \ \delta\left(\frac{m}{2}v^2\ \right) = \delta T$

Where

T= kinetic energy of the moving particle.

Equation (2) can be written as:

$\int_{t_1}^{t_2}(\ \delta T + \vec{F}.\,\delta\vec{r}\ )dt$

If $\vec{F}$ is conservative then there exists a scalar function $\phi(x, y, z)$, such that

$\vec{F}\ .\ \delta\vec{r}\ = d\phi$ or $\vec{F} = \overline{\nabla}\ \ \phi$ where the function ϕ is called the potential function and – ϕ is the potential energy of the particle in the field.

$\vec{F} = \overline{\nabla}\ \phi\ = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

And $F = xi + yj +\ zk$

$\therefore\ \delta F =\ \delta xi\ + \delta yj + \delta zk$

$\therefore\ \vec{F}.\delta\vec{r} = \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z = \delta\phi$

And therefore the Hamilton's equation can be written as:

$$\int_{t_1}^{t_2}(\ \delta T + \delta\phi)dt = 0\ or \quad \delta\int_{t_1}^{t_2}(\ T + \phi)dt = 0$$

Finally since $\phi = \ -V =$ the potential energy of the system.

$\delta\int_{t_1}^{t_2}(T - V)dt = 0$ Hamilton principle for a single mass particle in a conservative field.

2.3 Extended Hamilton's Principle:

From d'Alembert's principle which states that

$$\sum_{i=1}^{N}(F_i - m_i\ddot{r}_i).\delta r_i = 0 \ldots\ldots..(1)$$

Where N is number of particles.

Beginning with case in which the position vectors $F_i(\, i = 1,2, \ldots N)$ are all independent. From equation (1):

$\sum_{i=1}^{N} F_i - \delta\vec{r}_i = \delta W$ ………(2)

Where $\delta W \ldots. Virtula\, wowk$ of all applied forces including both conservative and non conservative forces. And if we reduce the second term in equation (1) to a form.

$$\frac{d}{dt}(m_i\dot{r}_i.\delta\vec{r}) = m_i\ddot{r}_i\,\delta r_i + m_i\dot{r}_i\delta\dot{r}_i = m_i\ddot{r}_i\,\delta r_i + \delta(\tfrac{1}{2}m_i\dot{r}_i)^2$$

But $\delta(\frac{1}{2}m_i\dot{r}_i)^2 = \delta T_i$ so

$$-m_i\ddot{r}_i\,\delta r_i = \delta T_i - \frac{d}{dt}(m_i\dot{r}_i.\delta\vec{r})$$

Then integrating w.r.t over the intervals.

$$-\int_{t_1}^{t_2} m_i\ddot{r}_i\,\delta r_i\,dt = \int_{t_1}^{t_2}\delta T_i\,dt - \int_{t_1}^{t_2}\frac{d}{dt}(m_i\dot{r}_i.\delta\vec{r})\,dt$$

Or

$$-\int_{t_1}^{t_2} m_i\ddot{r}_i\,\delta r_i\,dt = \int_{t_1}^{t_2}\delta T_i\,dt - \overset{0}{\cancel{m\dot{r}\delta r}}\Big|_{t_1}^{t_2}$$

But the virtual displacements are arbitrary

$\delta\vec{r_i} = 0$ at t =t_1 and t =t_2

$$\therefore\ -\int_{t_1}^{t_2} m_i \ddot{r}_i\, \delta r_i\, dt = \int_{t_1}^{t_2} \delta T_i\, dt$$

Summing up over i and integrating w.r.t , the second term of equation (1) becomes.

$-\int_{t_1}^{t_2} \sum_{i=1}^{N} m_i\, \ddot{r}_i\, \delta r_i\, dt = \int_{t_1}^{t_2} \delta T\, dt$ …… (3)

from equations 1, 2 .and 3

$\int_{t_1}^{t_2} (\delta T + \delta W) dt = 0 \ldots \ldots ..(4)$

$\delta \vec{r}_i = 0$ i=1,2,….N at at t =t_1 and t =t_2

Equation (4) Extended Hamilton's principle.

It is often convenient to divided the virtual work δW into two parts i.e.

$\delta W = \delta W_c + \delta W_{nc}$

Where

$\delta W_c \ldots ..virtual\ due\ to\ consevative\ force$

$\delta W_{nc} \ldots ..virtual\ due\ to\ non\ consevative\ force$

But

$\delta W_c = -\delta V$

V… the potential energy.

Then the Extended Hamilton's principle can be written as:

$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc}) dt = 0$

2.5 Lagrange's Equation:

Potential and Kinetic Energy in Generalized Coordinates:

Considering a system with *n* degrees of freedom, generalized coordinates refer to any set of *independent* coordinates equal in number to the *n* degrees of freedom of the system under consideration. , the generalized coordinates are denoted by *qi, i* = 1, 2, . . ., *n* and are used to express the scalar notion of kinetic energy *T* and potential energy *U*.

Potential energy *U* of a mechanical or flexible structural system typically only depends on the position of the system. Kinetic energy *T* typically depends on velocity, but may be also be position dependent. In terms of generalized coordinates *qi, i* = 1, 2, . . ., *n*, the scalar notion of kinetic energy *T* and potential energy *U* can be expressed as functions

$$\begin{aligned} T &= T(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) \\ U &= U(q_1, \ldots, q_n) \end{aligned} \tag{1}$$

that depend on the generalized positions q_i and generalized velocity $\dot{q}_i$ for $i = 1, 2, \ldots, n$.

Derivation of Lagrange's Equations

Considering an conservative system, where all external and internal forces have a potential. In that case, the sum of kinetic energy T and potential energy U will be constant and the differential is equal to zero:

$$d(T + U) = 0 \tag{2}$$

The above equation is basically a statement of the principle of conservation of energy. With the kinetic energy T and the potential energy U written as in (1), Lagrange's equations can be derived by summing up the kinetic and potential energy over all *generalized coordinates* q_i, $i = 1, 2, \ldots, n$.

With T and U given in (1) it is easy to see that

$$dU := \sum_{i=1}^{n} \frac{\partial}{\partial q_i} U(q_1, \ldots, q_n) dq_i \tag{3}$$

and

$$dT := \sum_{i=1}^{n} \frac{\partial}{\partial q_i} T(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) dq_i + \sum_{i=1}^{n} \frac{\partial}{\partial \dot{q}_i} T(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) d\dot{q}_i \tag{4}$$

In the remainder of the derivation, the arguments q_i and $\dot{q}_i$ of $U(\cdot)$ and $T(\cdot)$ are dropped for brevity.

The second term in dT depends on perturbations $d\dot{q}_i$ (the generalized velocity) and can be eliminated by considering the equation for kinetic energy ($\frac{1}{2}mv^2$) in generalized coordinates

$$T = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{q}_i \dot{q}_j \tag{5}$$

where m_{ij} denote the coefficients of the mass matrix in generalized coordinates.

discussed is deferred to later in this document. For now it suffices to know that $m_{ij} = m_{ji}$ and differentiation of T with respect to $\dot{q}_i$ gives

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{n} m_{ij} \dot{q}_j, \quad i = 1, 2, \ldots, n$$

The result can be back substituted into the expression for the kinetic energy T in (5) to obtain

$$T = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} \dot{q}_i$$

The second term with $d\dot{q}_i$ can be eliminated from (4) using the product rule:

$$2dT = \sum_{i=1}^{n} d\left(\frac{\partial T}{\partial \dot{q}_i}\right) \dot{q}_i + \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} d\dot{q}_i$$

and subtraction of (4) from the above equation yields

$$dT = \sum_{i=1}^{n} d\left(\frac{\partial T}{\partial \dot{q}_i}\right) \dot{q}_i - \sum_{i=1}^{n} \frac{\partial T}{\partial q_i} dq_i$$

$$dT = \sum_{i=1}^{n} d\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i - \sum_{i=1}^{n}\frac{\partial T}{\partial q_i}dq_i$$

Further simplification of this expression is obtained by the fact that

$$d\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)dq_i$$

making

$$dT = \sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right]dq_i \tag{6}$$

With (3) and (6), the equation of conservation of energy (2) now becomes

$$d(T+U) = \sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i}\right]dq_i = 0$$

Since q_i denote the generalized coordinates that are a set of *independent* coordinates, the above expression is satisfied if and only if

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0, \quad i = 1, 2, \ldots, n \tag{7}$$

Equation (7) constitutes Lagrange's equation for a conservative system, where all external and internal forces have a potential. For systems that are non-conservative, Lagrange's equation in (7) can be generalized by including a non-zero right side term

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i, \quad i = 1, 2, \ldots, n} \tag{8}$$