where Q_i denotes the (generalized) forces.

It is clear that writing down Lagrange's equations requires the partial derivative of the scalar functions of the kinetic energy $T(q_i, \dot{q}_i)$ and potential energy $U(q_i)$ with respect to the generalized coordinates q_i and generalized velocity $\dot{q}_i$ for each $i = 1, 2, \ldots, n$. A short-hand version of Lagrange's equations in (7) and (8) can obtained by defining a single scalar Lagrange function

$$L(q_i, \dot{q}_i) := T(q_i, \dot{q}_i) - U(q_i) \tag{9}$$

and realizing that

$$\frac{\partial}{\partial \dot{q}_i} U(q_i) = 0$$

As a result, (8) can also be written as the Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i, \quad i = 1, 2, \ldots, n$$

where L is the Lagrangian defined in (9).

So equation (8) is the most popular form of Lagrange equation.

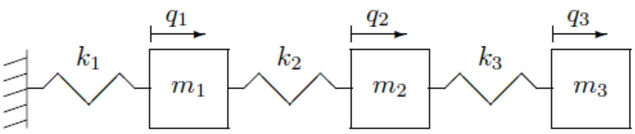
Application Procedure of Lagrange's method:

The application of Lagrange's method can be carried out as follows:

1. *Definition of the generalized coordinates q_i*
2. *Formulation of the kinetic energy T*
3. *Formulation of the potential energy U*
4. *Formulation of the generalized forces Qi (in case of forced vibration)*

EX1.2:

A free vibration of three degree of freedom system shown below, if it consist of three masses connected by three spring , formulate the system equation of motion.



Solution:

Take the coordinates the three displacements, q_1, q_2, q_3.

The kinetic energy of the system:

$$T = \frac{1}{2} m_1 \dot{q}_1{}^2 + \frac{1}{2} m_2 \dot{q}_2{}^2 + \frac{1}{2} m_3 \dot{q}_3{}^2$$

The Potential energy is

$$U = \frac{1}{2} k_1 q_1^2 + \frac{1}{2} k_2 (q_2 - q_1)^2 + \frac{1}{2} k_3 (q_3 - q_2)^2$$

Diff. the expressions for kinetic energy successively w.r.to velocities it yield:

$$\frac{\partial T}{\partial \dot{q}_1} = m_1 \dot{q}_1 \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = m_1 \ddot{q}_1$$

The kinetic energy is not a function of displacement so

$$\frac{\partial T}{\partial q_1} = 0$$

And

$$\frac{\partial T}{\partial \dot{q}_2} = m_2 \dot{q}_2 \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = m_2 \ddot{q}_2$$

$$\frac{\partial T}{\partial q_2} = 0$$

$$\frac{\partial T}{\partial \dot{q}_3} = m_3 \dot{q}_3 \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_3}\right) = m_3 \ddot{q}_3$$

$$\frac{\partial T}{\partial q_3} = 0$$

For the potential energy the same procedure.

$$\frac{\partial U}{\partial q_1} = (k_1 + k_2) q_1 - k_2 q_2$$

$$\frac{\partial U}{\partial q_2} = -k_2 q_1 + (k_2 + k_3) q_2 - k_3 q_3$$

$$\frac{\partial U}{\partial q_3} = -k_3 q_2 + k_3 q_3$$

For free vibration $Q_i = 0$

Subs. The above derivations in equation (8) we get:

$$m_1 \ddot{q}_1 + (k_1 + k_2) q_1 - k_2 q_2 = 0$$

$$m_2 \ddot{q}_2 - k_2 q_1 + (k_2 + k_3) q_2 - k_3 q_3 = 0$$

$$m_3 \ddot{q}_3 - k_3 q_2 + k_3 q_3 = 0$$

The three Lagrange equations can be combined in matrix format:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3. Natural Frequencies and Mode Shapes:

3.1Eigen Value Problem:

Consider that we have an un damped free vibration system (*i.e* $\{F\} = 0$). The general equation of motion of which can be written as follows:

$[M]\{\ddot{X}\} + [k]\{X\} = \{0\}$ ………….. (1)

If the system given an initial displacement s or initial velocities or both for example.

$x_{i(t)} = X_i \, T_{(t)}$ …………….(2)

Where

X_i = constant (Amplitude)

$T_{(t)}$ = $f(t)$

And if we write

$$\{\Phi\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \text{mode shape.}$$

Equation (1) become

$[M]\{\Phi\}\ddot{T}_{(t)} + [K]\{\Phi\}T_{(t)} = \{0\}$ ……….. (3)

Equation (3) can be written as scalier:

$\left(\sum_{j=1}^{n} m_{ij}X_j\right)\ddot{T}_{(t)} + \left(\sum_{j=1}^{n} k_{ij}X_j\right)T_{(t)} = 0$ i=1,2,3 ….,n

Or

$$-\frac{\ddot{T}(t)}{T(t)} = \frac{\sum_{j=1}^{n} k_{ij}X_j}{\sum_{j=1}^{n} m_{ij}X_j} = constant = \omega^2$$

Or

$$\sum_{j=1}^{n} (\, k_{ij} - \omega^2 m_{ij}) \, X_j = 0$$

And in matrix form

$\big[[k] - \omega^2[m]\big]\{\Phi\} = \{0\}$ …………. (4)

Equation (4) is called characteristic equation .

ω = Eigen value = natural frequency.

The solution of the characteristic equation can be expressed as:

Let $\lambda = \frac{1}{\omega^2}$

So

Equation (4) can be written as

$\big[\lambda[k] - [m]\big]\{x\} = \{0\}$ ……….. (5)

Multiplying equation (5) by $[k]^{-1}$ gives

$\big[\lambda[I] - [k]^{-1}[m]\big]\{x\} = \{0\}$ …… (6)

Let $[k]^{-1}[m] = [D]$

Where

$[D]$ ….. Dynamic matrix.

So

$\lambda[I]\{x\} = [D]\{x\}$, $[I]$… Is the identily matrix

and for nontrivial solution of $\{x\}$ the determinant must be zero.

So,

$\Delta = |\lambda[I] - [D]| = 0$ ……. (7)

Equation (7) called the frequency equation. In this case a numerical methods should be used to find the roots of equation.

<u>**EX:3.1**</u>

Find the natural frequency and mode shapes of the system shown in Ex:1.1 for $k_1 = k_2 = k_3 = k$, $m_1 = m_2 = m_3 = m$

<u>Solution:</u>

From Ex:1.1

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{k_1} & \frac{1}{k_1} & \frac{1}{k_1} \\ \frac{1}{k_1} & \left(\frac{1}{k_1}+\frac{1}{k_2}\right) & \left(\frac{1}{k_1}+\frac{1}{k_2}\right) \\ \frac{1}{k_1} & \left(\frac{1}{k_1}+\frac{1}{k_2}\right) & \left(\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}\right) \end{bmatrix}}_{[a]} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$[a] = \frac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$[m] = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[D] = [k]^{-1}[m] = [a][m]$$

So,

$$[D] = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\Delta = |\lambda[I] - [D]| = \left|\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\right| = 0 \quad , \ \lambda = \frac{1}{\omega^2}$$

Dividing by λ, and setting $\alpha = \frac{m}{\lambda k} = \frac{m\omega^2}{k}$

One can find that

$$\begin{vmatrix} 1-\alpha & -\alpha & -\alpha \\ -\alpha & 1-2\alpha & -2\alpha \\ -\alpha & -2\alpha & 1-3\alpha \end{vmatrix} = \alpha^3 - 5\alpha^2 + 6\alpha - 1 = 0$$

The roots of cubic equation are:

$$\alpha_1 = \frac{m\omega_1^2}{k} = 0.198 \ \Rightarrow \omega_1 = 0.445\sqrt{\frac{k}{m}}$$

$\alpha_2 = \frac{m\omega_2^2}{k} = 1.555 \quad \Rightarrow \omega_2 = 1.247\sqrt{\frac{k}{m}}$

$\alpha_3 = \frac{m\omega_3^2}{k} = 3.249 \quad \Rightarrow \omega_3 = 1.8025\sqrt{\frac{k}{m}}$

Then the mode shapes (Eigen vector). Can be obtained from equation (6)

$\left[\lambda[I] - [k]^{-1}[m]\right]\{x\}^{(i)} = \{0\}$

Where $\{x\}^{(i)}$ denotes the i^{th} mode shape.

<u>First mode</u>

$\{x\}^{(i)} = \begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{Bmatrix}$, $\lambda_1 = \frac{1}{\omega_1^2} = 5.049\frac{m}{k}$

Substitute in equation (7) it gives:

$$\left[5.049\frac{m}{k}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\right]\begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\frac{m}{k}\begin{bmatrix} 4.049 & -1 & -1 \\ -1 & 3.049 & -2 \\ -1 & -2 & 2.049 \end{bmatrix}\begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

From which

$x_2^{(1)} + x_3^{(1)} = 4.049\, x_1^{(1)} \quad \dots \dots (1)$

$3.049\, x_2^{(1)} - 2x_3^{(1)} = x_1^{(1)} \dots \dots (2)$

$-2\, x_2^{(1)} + 2.049\, x_3^{(1)} = x_1^{(1)} \dots\dots (3)$

We have three equations with three unknowns.

So,

$x_2^{(1)} = 1.8019\ x_1^{(1)}\quad , x_3^{(1)} = 2.247\, x_1^{(1)}$

Thus the first mode shape is given by

$$\{x\}^{(1)} = x_1^{(1)} \begin{Bmatrix} 1 \\ 1.8019 \\ 2.247 \end{Bmatrix}$$

Where the value of $x_1^{(1)}$ $can\ be\ chosen$ arbitrarily, and in the same procedure :

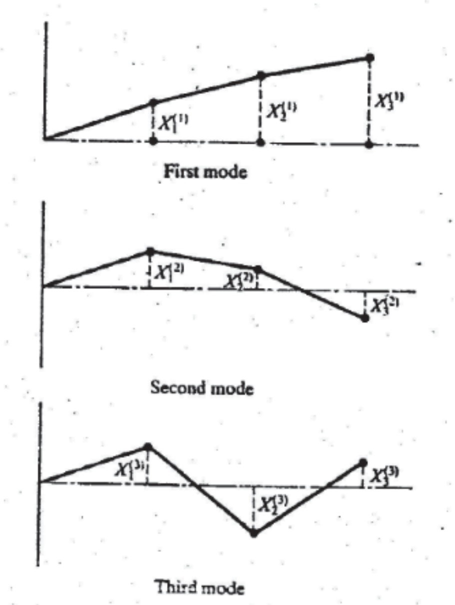
<u>Second mode</u>

$$\{x\}^{(2)} = x_1^{(2)} \begin{Bmatrix} 1 \\ 0.445 \\ -0.8020 \end{Bmatrix}$$

<u>Third mode</u>

$$\{x\}^{(3)} = x_1^{(3)} \begin{Bmatrix} 1 \\ -1.2468 \\ 0.5544 \end{Bmatrix}$$

The mode shapes graph is shown below.



3.2 Orthogonality of Eigen vector (Normal modes):

The Orthogonality or normal mode is an efficient method for the calculation of the natural frequency and mode shapes, where they are can be orthogonal with respect to mass and stiffness matrix as shown below.

Let ω_i is the ith natural frequency and$\{\phi\}^{(i)}$ is corresponding normal mode where satisfied the equation.

$$\big[[k] - \omega^2[m]\big]\{\Phi\} = \{0\}$$

$$\therefore\ \omega_i^2[m]\{\Phi\}^{(i)} = [k]\{\Phi\}^{(i)} \quad \ldots\ldots (1)$$

And

$$\omega_j^2[m]\{\Phi\}^{(j)} = [k]\{\Phi\}^{(j)} \quad \ldots\ldots.(2)$$

By multiplying equations (1) & (2) by $\{\Phi\}^{(j)T}, \{\Phi\}^{(i)T}$ $respectively$, <u>where $[k]$ and $[m]$ are symmetry matrices</u> .

$$\omega_i^2\{\Phi\}^{(j)T}[m]\{\Phi\}^{(i)} = \{\Phi\}^{(j)T}[k]\{\Phi\}^{(i)} \equiv \{\Phi\}^{(i)T}[k]\{\Phi\}^{(j)} \quad \ldots.. (3)$$

$$\omega_j^2\{\Phi\}^{(i)T}[m]\{\Phi\}^{(j)} = \{\Phi\}^{(i)T}[k]\{\Phi\}^{(j)} \equiv \{\Phi\}^{(i)T}[k]\{\Phi\}^{(j)} \quad \ldots\ldots(4)$$

Subtracting equation (4) from (3)

$(\omega_i^2 - \omega_j^2)\ \{\ \Phi\}^{(j)T}[m]\{\ \Phi\}^{(i)} = 0$ …….. (5)

Since $\omega_i \neq \omega_j$ so

$\{\ \Phi\}^{(j)T}[m]\{\ \Phi\}^{(i)} = 0$ ……… (6) i≠ j (eigenvector or orthogonal with respect to mass matrix)

And

$\{\ \Phi\}^{(j)T}[k]\{\ \Phi\}^{(i)} = 0$ ……… (7) i≠ j (eigenvector or orthogonal with respect to stiffness matrix)

If (**i=j**) then equation (6) will not equal zero, it will equal to generalized mass matrix $\mathbf{M_{ii}}$ of the i^{th} mode.

or

$\mathbf{M_{ii}} = \{\ \Phi\}^{(i)T}[m]\{\ \Phi\}^{(i)}$ …… (8)

And equation (7) become equal to generalized stiffness matrix $\mathbf{K_{ii}}$ of the i^{th} mode.

$\mathbf{K_{ii}} = \{\ \Phi\}^{(i)T}[k]\{\ \Phi\}^{(i)}$ ……..(9)

If each of the normal mode $\{\ \Phi\}^{(i)}$ *is divided* by square root of the generalized mass M_{ii} (i.e $\frac{\{\ \Phi\}^{(i)}}{\sqrt{\mathrm{Mii}}}$), then equation (8) will be equal to unity, and equation (9) will be equal to λ_i .

So,

$\{\ \tilde{\Phi}\}^{(i)T}[m]\{\tilde{\Phi}\}^{(i)} = 1$

$\{\tilde{\Phi}\ \}^{(i)T}[k]\{\tilde{\Phi}\}^{(i)} = \lambda_i$

$\{\tilde{\Phi}\}^{(i)}$ **…….** Orthonormal mode or weighted normal mode.

3.3 Modal Matrix P

The modal matrix form is an a square matrix consist of n normal mode (eigenvectors) as follows:

Let us take a three DOF system.

$$[P] = \left[\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}^{(1)} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}^{(2)} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}^{(3)} \right] = (\phi_1 \quad \phi_2 \quad \phi_3)$$

And

$$[P]^T = \begin{bmatrix} (x_1 & x_2 & x_3)^{(1)} \\ (x_1 & x_2 & x_3)^{(2)} \\ (x_1 & x_2 & x_3)^{(3)} \end{bmatrix} = (\phi_1 \quad \phi_2 \quad \phi_3)^T$$

Now

$$[P]^T[m][P] = (\phi_1 \quad \phi_2 \quad \phi_3)^T[m](\phi_1 \quad \phi_2 \quad \phi_3)$$

$$= \begin{bmatrix} \phi_1^T[m]\phi_1 & \phi_1^T[m]\phi_2 & \phi_1^T[m]\phi_3 \\ \phi_2^T[m]\phi_1 & \phi_2^T[m]\phi_2 & \phi_2^T[m]\phi_3 \\ \phi_3^T[m]\phi_1 & \phi_3^T[m]\phi_2 & \phi_3^T[m]\phi_3 \end{bmatrix} = \begin{bmatrix} M_{11} & 0 & 0 \\ 0 & M_{22} & 0 \\ 0 & 0 & M_{33} \end{bmatrix}$$

Where

M_{ii} …… is generalized mass

Similarly for K_{ii}

$$[P]^T[k][P] = \begin{bmatrix} K_{11} & 0 & 0 \\ 0 & K_{22} & 0 \\ 0 & 0 & K_{33} \end{bmatrix}$$

If we replaced ϕ_i in $[P]$ matrix the orthonormal mode by $\widetilde{\Phi_i}$, the modal matrix is $\tilde{P}$

Or

$\widetilde{P^T}[m]\tilde{P} = [I]$

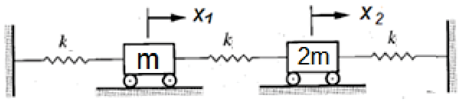
$\widetilde{P^T}[k]\tilde{P} = \Lambda$

Where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad , \ \lambda = \omega^2$$

EX:3.2

Determine the natural frequency of the system shown in figure below if.



$$\Phi_1 = \begin{Bmatrix} 0.731 \\ 1.00 \end{Bmatrix}$$

$$\Phi_2 = \begin{Bmatrix} -2.73 \\ 1.00 \end{Bmatrix}$$

Solution:

The mass and stiffness matrix are:

$$[m] = m \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad , \quad [k] = k \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[P] = \begin{bmatrix} 0.731 & -2.73 \\ 1.00 & 1.00 \end{bmatrix}$$

$$[P]^T[m][P] = \begin{bmatrix} 0.731 & 1.00 \\ -2.73 & 1.00 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{bmatrix} 0.731 & -2.73 \\ 1.00 & 1.00 \end{bmatrix}$$

$$= \begin{bmatrix} 2.53m & 0 \\ 0 & 9.45m \end{bmatrix} = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}$$

$$[\tilde{P}] = \begin{bmatrix} \frac{0.731}{\sqrt{2.53m}} & \frac{-2.73}{\sqrt{9.45m}} \\ \frac{1}{\sqrt{2.53m}} & \frac{1}{\sqrt{9.45m}} \end{bmatrix} = \begin{bmatrix} \frac{0.459}{\sqrt{m}} & \frac{-0.888}{\sqrt{m}} \\ \frac{0.628}{\sqrt{m}} & \frac{0.325}{\sqrt{m}} \end{bmatrix}$$

$$\widetilde{P^T}[m]\tilde{P} = [I]$$

And

$$\widetilde{P^T}[k]\tilde{P} = \begin{bmatrix} 0.635\frac{k}{m} & 0 \\ 0 & 2.365\frac{k}{m} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

So

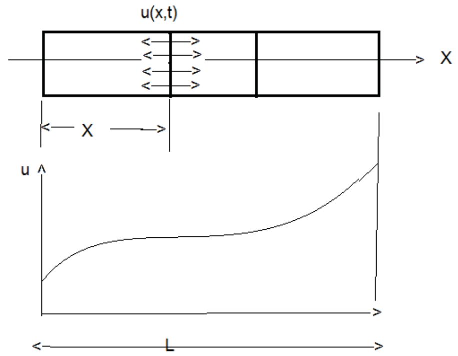
$$\omega_1 = 0.7968\sqrt{\frac{k}{m}} \;, \quad \omega_2 = 1.3578\sqrt{\frac{k}{m}}$$

4.Vibrations of continuous systems.

Introduction:

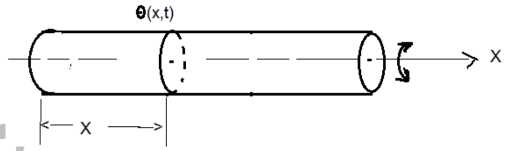
In this study the body is no longer treat as a rigid, so elastic body vibrations will be investigate, for such systems an infinite degree of freedom are expected. Three simple cases of vibration then will be analysis, which they vibrations of bars , vibrations of shafts, and vibrations of beams, under longitudinal (axial) oscillation, torional oscillation, and transverse oscillation respectively. As follows:

Let consider a bar shown in figure below:



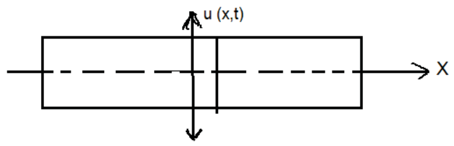
Longitudinal oscillation
Of a Bar

And for a shafts:



Torsional oscillation of a shaft.

And for the beam:



Transverse oscillation

In all above cases the assumptions are:
- Uniform
-Small oscillation
-free vibration
- Undamped

4.1Longitudinal (Axial) vibration of uniform bars:

Consider a uniform bar as shown in figure 2.1 and an element of the bar a distance x from one end. When vibrating the forces on the element and the deflection (u) of the element at some time t are as shown:

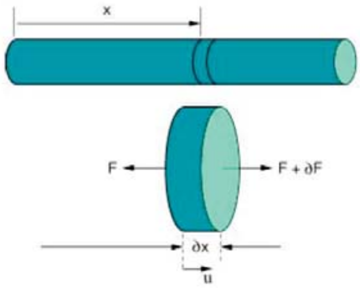


Fig.(2.1)

For the element the stress/strain equation gives

$$F = AE\frac{\partial u}{\partial x} \ldots\ldots\ldots\ldots\ldots (1)$$

where A is the cross-section area and E is the elastic (Young's) modulus.

Applying Newton 2nd law to the element.

$$\sum F = ma$$

$$F + \partial F - F = m\frac{\partial^2 u}{\partial t^2}$$

and if ρ is the density, the mass of the element

$$m = \rho A \partial x$$

so that,

$$\partial F = \rho A\, \partial x \frac{\partial^2 u}{\partial t^2}$$

so that

$$\frac{\partial F}{\partial x} = \rho A \frac{\partial^2 u}{\partial t^2} \quad \ldots\ldots (2)$$

Substituting for F from eqn. (1) in eqn. (2)

$$\frac{\partial}{\partial x} AE \frac{\partial u}{\partial x} = \rho A \frac{\partial^2 u}{\partial t^2}$$

and rearranging

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{E} \frac{\partial^2 u}{\partial t^2} \quad \ldots\ldots\ldots\ldots (3)$$

To find the solution of equation (3) let us take a sinusoidal motion (steady – state) assume that.

$$u(x, t) = U(x) e^{i\omega t}$$

From which

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 U(x) e^{i\omega t}$$

And

$$\frac{\partial^2 u}{\partial x^2} = \ddot{U}_{(x)}\ e^{i\omega t}$$

Substituting in equation (3) it gives:

$$\ddot{U}_{(x)} e^{i\omega t} = -\frac{\rho\omega^2}{E} U(x) e^{i\omega t}$$

Thus

$$\ddot{U}_{(x)} + \frac{\rho\omega^2}{E}U(x) = 0 \quad \ldots\ldots\ldots\ldots\ldots (4)$$

Let the solution of equation (4) is

$$U(x) = Be^{\beta_1 x} + Ce^{\beta_2 x}$$

so that to find β_1 and β_2 we substitute $U(x) = Be^{\beta x}$ and find two values for β.

Thus

$$\beta^2 Be^{\beta x} + \frac{\rho\omega^2}{E}Be^{\beta x} = 0$$

$$\text{and } \beta^2 = -\frac{\rho\omega^2}{E}$$

Let $\lambda^2 = \dfrac{\rho\omega^2}{E}$ so that

$$\beta^2 = -\lambda^2$$

$$\therefore \beta = \pm i\lambda$$

Hence

$$\boxed{U(x) = Be^{i\lambda x} + Ce^{-i\lambda x} \quad \ldots\ldots\ldots\ldots\ldots (5)}$$

The constant B , and C depend and can be found from the boundary conditions of the bar.

EX:4.1

A free-free end bar find it natural frequencies and mode shapes equations .

From table(1) or there is no force at either ends from equation (1)

$$F = AE\frac{\partial u}{\partial x} \quad \text{so that when there is no force} \quad \frac{\partial u}{\partial x} = 0$$

From equation (5)

$$\frac{\partial u}{\partial x} = i\lambda B e^{i\lambda x} - i\lambda C e^{-i\lambda x} = 0$$

so that when x=0 $\qquad B - C = 0 \quad \ldots\ldots\ldots \quad (a)$

and when x=L $\qquad Be^{i\lambda L} - Ce^{-i\lambda L} = 0 \quad \ldots\ldots\ldots \quad (b)$

From (a) $C = B$ and substituting in (b) $\qquad B\left(e^{i\lambda L} - e^{-i\lambda L}\right) = 0$

Thus either B=0 and there is no vibration or B may exist and there will be vibration without any continuing excitation (ie at a natural frequency) if

$$e^{i\lambda L} - e^{-i\lambda L} = 0$$

$$\text{ie when } \cos\lambda L + i\sin\lambda L - \cos\lambda L + i\sin\lambda L = 2i\sin\lambda L = 0$$

The natural frequencies are thus when $\sin\lambda L = 0$ which is when $\lambda L = n\pi$ and $n = 0 \to \infty$. Substituting for

$$\lambda = \omega\sqrt{\frac{\rho}{E}}$$

$$\omega\sqrt{\frac{\rho}{E}}L = n\pi \quad \text{and hence} \quad \omega_n = \frac{n\pi}{L}\sqrt{\frac{E}{\rho}} \qquad n = 0 \to \infty$$

As a continuous bar has an infinite number of degrees-of-freedom we should not be surprised to find that it has an infinite number of natural frequencies.

To find the mode shapes we return to equation (5) with C= B

$$U(x) = B\left(e^{i\lambda x} + e^{-i\lambda x}\right) = 2B\cos\lambda x$$

And substituting $\lambda L = n\pi$ (ie the natural frequencies)

$$U(x) = A\cos\left(n\pi\frac{x}{L}\right)$$

As the bar is free/free we have a zero frequency mode when n=0 and $U(x) = A$ which is a solid body motion.

Example:4.2 free/free(Axial force)

From the diagram above it should be noted that at the right hand end when x = L (the length of the bar) the force F is in the same positive direction as u.

Thus for x = L from (1) $\frac{\partial u(L)}{\partial x} = \frac{F}{AE}$

However when x = 0 the force F is in the negative u direction

Thus for x =0 from (1) $\frac{\partial u(0)}{\partial x} = -\frac{F}{AE}$

Response when excitation is at x = L

For this case when x=L $F = F_L e^{i\omega t}$ and when x=0 F=0. The response is $u(x) = U(x)e^{i\omega t}$

When x=0 from equation (5) $\frac{\partial u}{\partial x} = i\lambda Be^{i\lambda x} - i\lambda Ce^{-i\lambda x} = 0$

so that when x=0 $B - C = 0$ and hence $B = C$

When x = L, $F = F_L e^{i\omega t}$ and $\frac{\partial u(L)}{\partial x} = U'(L)e^{i\omega t} = \frac{F_L e^{i\omega t}}{AE}$

therefore $i\lambda Be^{i\lambda L} - i\lambda Ce^{-i\lambda L} = \frac{F_L}{AE}$

Put C = B so that $B(i\lambda\cos\lambda L - \lambda\sin\lambda L - i\lambda\cos\lambda L - \lambda\sin\lambda L) = \frac{F_L}{AE}$

Therefore,

$$B = C = \frac{-F_L}{2AE\lambda\sin\lambda L} \quad \text{..} \quad (6)$$

returning to equation (5)

$$U(x) = Be^{i\lambda x} + Ce^{-i\lambda x}$$

and substituting for B and C from (6)

$$U(x) = \left(e^{i\lambda x} + e^{-i\lambda x}\right)\frac{-F_L}{2AE\lambda \sin\lambda L}$$

$$\therefore\ U(x) = \left(\cos\lambda x + i\sin\lambda x + \cos\lambda x - i\sin\lambda x\right)\frac{-F_L}{2AE\lambda \sin\lambda L}$$

so that

$$U(x) = -\frac{F_L \cos\lambda x}{AE\lambda \sin\lambda L}$$

and hence the response at a position x along the bar when excited at x=L is

$$\frac{U(x)}{F_L} = -\frac{\cos\lambda x}{AE\lambda \sin\lambda L} \quad \text{..............................} \quad (7)$$

for the case when x = L we obtain the response at the excitation position,

$$\frac{U(L)}{F_L} = -\frac{\cos\lambda L}{AE\lambda \sin\lambda L} \quad \text{..............................} \quad (8)$$

The major point to note is that the responses have $\cos\lambda L$ in the denominator. Thus the response will tend to infinity and resonance will occur when $\cos\lambda L = 0$ which is when $\lambda L = \frac{2n-1}{2}\pi$ and $n = 1 \rightarrow \infty$.

It should be noted that there are anti-resonances. If we consider the response at the end of the bar,

$$\frac{u(L)}{F_L} = -\frac{\cos\lambda L}{AE\lambda \sin\lambda L}$$

it is apparent that which is when $\lambda L = \frac{2n-1}{2}\pi$ and $n = 1 \rightarrow \infty$.

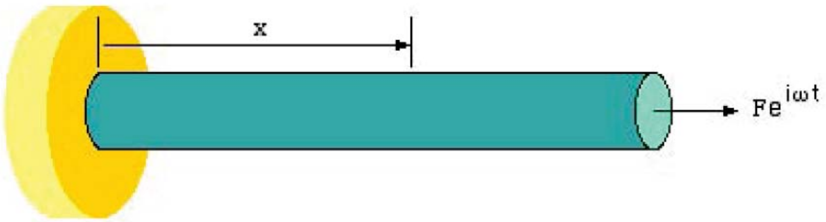
Substituting for

$$\lambda = \omega\sqrt{\frac{\rho}{E}}$$

$$\omega\sqrt{\frac{\rho}{E}}L = \frac{2n-1}{2}\pi \text{ and hence } \omega = \frac{(2n-1)\pi}{2L}\sqrt{\frac{E}{\rho}} \qquad n = 1 \rightarrow \infty$$

EX4.3 Clamped/free (Axial force)

Clamped/free bar



B.C

At fixed (clamped) end u(0,t) = 0 (from table 1)

For sinusoidal excitation and response we have shown,

$$U(x) = Be^{i\lambda x} + Ce^{-i\lambda x} \quad \text{...} \quad (1)$$

Therefore when $x = 0$ $U(0) = 0 = B + C$ so that $C = -B$

as for the free/free case

When $x = L$, $F = F_L e^{i\omega t}$ and $\dfrac{\partial u(L)}{\partial x} = U'(L)e^{i\omega t} = \dfrac{F_L e^{i\omega t}}{AE}$

therefore

$$i\lambda Be^{i\lambda L} - i\lambda Ce^{-i\lambda L} = \frac{F_L}{AE}$$

and substituting $C = -B$

$$i\lambda Be^{i\lambda L} + i\lambda Be^{-i\lambda L} = \frac{F_L}{AE}$$

Therefore,

$$B = -C = \frac{F_L}{2\,AE\cos\lambda L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

subst. in eqn. (1)

$$U(x) = \left(\frac{e^{i\lambda x} - e^{-i\lambda x}}{2i}\right)\frac{F_L}{2\,AE\cos\lambda L} = \frac{F_L\sin\lambda x}{2\,AE\cos\lambda L}$$

so that

$$U(x) = \frac{F_L\sin\lambda x}{2\,AE\cos\lambda L}$$

and hence the response at a position x along the bar when excited at x=L is

$$\frac{U(x)}{F_L} = \frac{\sin\lambda x}{2\,AE\cos\lambda L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

for the case when x = L we obtain the response at the excitation position,

$$\frac{U(L)}{F_L} = \frac{\sin\lambda L}{2\,AE\cos\lambda L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

The major point to note is that the responses have $\cos\lambda L$ in the denominator. Thus the response will tend to infinity and resonance will occur when $\cos\lambda L = 0$ which is when $\lambda L = \frac{2n-1}{2}\pi$ and $n = 1 \rightarrow \infty$.

Substituting for

$$\lambda = \omega\sqrt{\frac{\rho}{E}}$$

$$\omega\sqrt{\frac{\rho}{E}}L = \frac{2n-1}{2}\pi \text{ and hence } \omega = \frac{(2n-1)\pi}{2L}\sqrt{\frac{E}{\rho}} \quad n = 1 \rightarrow \infty$$