4.2Torsional Vibrations of uniform bars:

For torsional vibration very similar maths applies to that developed for axial vibration. For uniform bars there are only four parameter differences, E is replaced throughout by G; A by J; F by T and u by θ.

The receptances are then the same as for the axial case but with $\lambda^2 = \rho\omega^2 / G$ thus for

Free/free

$$\frac{\Theta(x)}{T_L} = -\frac{\cos\lambda x}{GJ\lambda\sin\lambda L} \quad \text{.......} \quad (1)$$

$$\frac{\Theta(L)}{T_L} = -\frac{\cos\lambda L}{GJ\lambda\sin\lambda L} \quad \text{.......} \quad (2)$$

$$\frac{\Theta(0)}{T_L} = -\frac{1}{GJ\lambda\sin\lambda L} \quad \text{.......} \quad (3)$$

free/free (torsional)

The natural frequencies are thus when $\sin\lambda L = 0$ which is when $\lambda L = n\pi$ and $n = 0 \rightarrow \infty$. Substituting for

$$\lambda = \omega\sqrt{\frac{\rho}{G}}$$

$$\omega\sqrt{\frac{\rho}{G}}L = n\pi \quad \text{and hence} \quad \omega_n = \frac{n\pi}{L}\sqrt{\frac{G}{\rho}} \qquad n = 0 \rightarrow \infty$$

<u>Clamped/free</u> **(Torsional forced)**

$$\frac{\Theta(x)}{T_L} = \frac{\sin\lambda x}{GJ\lambda\cos\lambda L}$$

$$\frac{\Theta(L)}{T_L} = \frac{\sin\lambda L}{AE\lambda\cos\lambda L}$$

The natural frequencies are thus when $\cos\lambda L = 0$ which is when $\lambda L = \frac{2n-1}{2}\pi$ and $n = 1 \rightarrow \infty$.

Substituting for

$$\lambda = \omega\sqrt{\frac{\rho}{G}}$$

$$\omega\sqrt{\frac{\rho}{G}}L = \frac{2n-1}{2}\pi \text{ and hence } \omega = \frac{(2n-1)\pi}{2L}\sqrt{\frac{G}{\rho}} \quad n = 1 \rightarrow \infty$$

Boundary condition	At left end ($x = 0$)	At right end ($x = l$)
Fixed end	$\theta(0, t) = 0$ $x = 0$	$\theta(l, t) = 0$ $x = l$
Free end	$\frac{\partial\theta}{\partial x}(0, t) = 0$ $x = 0$	$\frac{\partial\theta}{\partial x}(l, t) = 0$ $x = l$
End torsional spring (spring constant = $\underset{\sim}{k}$)	$\underset{\sim}{k}_t$ $GJ\frac{\partial\theta}{\partial x}(0, t) = \underset{\sim}{k}_t\theta(0, t)$ $x = 0$	$\underset{\sim}{k}_t$ $GJ\frac{\partial\theta}{\partial x}(l, t) = -\underset{\sim}{k}_t\theta(l, t)$ $x = l$
End torsional damper (damping constant = $\underset{\sim}{c}$)	$\underset{\sim}{c}_t$ $GJ\frac{\partial\theta}{\partial x}(0, t) = \underset{\sim}{c}_t\frac{\partial\theta}{\partial t}(0, t)$ $x = 0$	$\underset{\sim}{c}_t$ $GJ\frac{\partial\theta}{\partial x}(l, t) = -\underset{\sim}{c}_t\frac{\partial\theta}{\partial t}(l, t)$ $x = l$
End inertia (inertia = $\underset{\sim}{I}_0$)	$\underset{\sim}{I}_0$ $GJ\frac{\partial\theta}{\partial x}(0, t) = \underset{\sim}{I}_0\frac{\partial^2\theta}{\partial t^2}(0, t)$ $x = 0$	$\underset{\sim}{I}_0$ $GJ\frac{\partial\theta}{\partial x}(l, t) = -\underset{\sim}{I}_0\frac{\partial^2\theta}{\partial t^2}(l, t)$ $x = l$

<u>Table (2) Boundary conditions for torsional</u>

<u>4.3Transverse vibration of bars:</u>

Consider a bar on transverse vibration the formulation of its equation of motion can be carried out as follows:

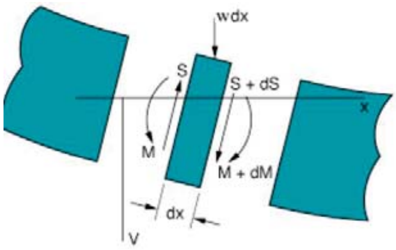


Figure (1)

The analysis to be presented is based upon the Bernoulli-Euler equations and ignores shear and rotary inertia effects. An element of a shaft is shown in above figure . Applying Newton 2nd law the **v** direction for small deflections

$$\sum F = ma$$

$$S - (S + dS) + wdx = \rho A dx \frac{\partial^2 v}{\partial t^2} \quad \text{.........} \quad (1)$$

Where
S …… Shear force.
ρ ……Material density.
A ……. c.s.a
V ……. Deflection of the bar from static.
W …….. weight per unit length.

Equation (1) reduced to

$$\frac{dS}{dx} + w = \rho A \frac{\partial^2 v}{\partial t^2} \quad \text{.............} \quad (2)$$

Since rotary inertia effects are ignored the sum of moments on the element are zero so that

$$\sum M = 0$$

$$Sdx + M + dM - M = 0$$

and therefore

$$S = -\frac{dM}{dx} \quad \text{..........................(3)}$$

From simple bending theory

$$M = EI\frac{\partial^2 v}{\partial x^2} \quad \text{..............................(4)}$$

where E is Young's modulus and I is the second moment of area of the section.
Substituting (4) in (3) gives

$$S = -EI\frac{\partial^3 v}{\partial x^3} \quad \text{..............................(5)}$$

and substituting this value for S in (2)

$$-EI\frac{\partial^4 v}{\partial x^4} + w = \rho A\frac{\partial^2 v}{\partial t^2}$$

and rearranging gives

$$\frac{\partial^2 v}{\partial t^2} + \frac{EI}{\rho A}\frac{\partial^4 v}{\partial x^4} = \frac{w}{\rho A} \quad \text{..........................(6)}$$

The right hand term results from gravity and results in a deflection from the weight of the bar. If v is taken to be the deflection from the static equilibrium position then the right hand term becomes zero so that,

$$\boxed{\frac{\partial^2 v}{\partial t^2} + \frac{EI}{\rho A}\frac{\partial^4 v}{\partial x^4} = 0 \quad \text{..............................(7)}}$$

Boundary condition	At left end ($x = 0$)	At right end ($x = l$)
Free end (bending moment = 0, shear force = 0)	$x = 0$ $EI\frac{\partial^2 w}{\partial x^2}(0,t)=0$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(0,t)}=0$	$x = l$ $EI\frac{\partial^2 w}{\partial x^2}(l,t)=0$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(l,t)}=0$
Fixed end (deflection = 0, slope = 0)	$x = 0$ $w(0,t)=0$ $\frac{\partial w}{\partial x}(0,t)=0$	$x = l$ $w(l,t)=0$ $\frac{\partial w}{\partial x}(l,t)=0$
Simply supported end (deflection = 0, bending moment = 0)	$x = 0$ $w(0,t)=0$ $EI\frac{\partial^2 w}{\partial x^2}(0,t)=0$	$x = l$ $w(l,t)=0$ $EI\frac{\partial^2 w}{\partial x^2}(l,t)=0$
Sliding end (slope = 0, shear force = 0)	$x = 0$ $\frac{\partial w}{\partial x}(0,t)=0$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(0,t)}=0$	$x = l$ $\frac{\partial w}{\partial x}(l,t)=0$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(l,t)}=0$
End spring (spring constant = $\underline{k}$)	$\underline{k}$ $x = 0$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(0,t)}= -\underline{k}\,w(0,t)$ $EI\frac{\partial^2 w}{\partial x^2}(0,t)=0$	$\underline{k}$ $x = l$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(l,t)}= +\underline{k}\,w(l,t)$ $EI\frac{\partial^2 w}{\partial x^2}(l,t)=0$
End damper (damping constant = $\underline{c}$)	$\underline{c}$ $x = 0$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(0,t)}= -\underline{c}\frac{\partial w}{\partial t}(0,t)$ $EI\frac{\partial^2 w}{\partial x^2}(0,t)=0$	$\underline{c}$ $x = l$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(l,t)}= +\underline{c}\frac{\partial w}{\partial t}(l,t)$ $EI\frac{\partial^2 w}{\partial x^2}(l,t)=0$
End mass (mass = $\underline{m}$ with negligible moment of inertia)	$\underline{m}$ $x = 0$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(0,t)}= -\underline{m}\frac{\partial^2 w}{\partial t^2}(0,t)$ $EI\frac{\partial^2 w}{\partial x^2}(0,t)=0$	$\underline{m}$ $x = l$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(l,t)}= +\underline{m}\frac{\partial^2 w}{\partial t^2}(l,t)$ $EI\frac{\partial^2 w}{\partial x^2}(l,t)=0$
End mass with moment of inertia (mass = $\underline{m}$, moment of inertia = I_0)	$\underline{m}, I_0$ $x = 0$ $EI\frac{\partial^2 w}{\partial x^2}(0,t)= I_0\frac{\partial^3 w}{\partial x \partial t^2}(0,t)$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(0,t)}= \underline{m}\frac{\partial^2 w}{\partial t^2}(0,t)$	$\underline{m}, I_0$ $x = l$ $EI\frac{\partial^2 w}{\partial x^2}(l,t)= -I_0\frac{\partial^3 w}{\partial x \partial t^2}(l,t)$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(l,t)}= -\underline{m}\frac{\partial^2 w}{\partial t^2}(l,t)$

Table (3) Boundary condition for the transverse vibration of beam.

Free Vibrations

Steady-state sinusoidal motion

For small amplitudes it may be assumed that $v(x,t) = V(x)e^{i\omega t}$ which is a sinusoidal vibration with an amplitude varying along the bar.

Substituting in (7)

$$-\omega^2 V(x) + \frac{EI}{\rho A}\frac{\partial^4 V(x)}{\partial x^4} = 0$$

$$\text{thus} \quad \frac{\partial^4 V(x)}{\partial x^4} - \frac{\rho A \omega^2}{EI} V(x) = 0$$

and the general form of the solution is

$$\boxed{V(x) = A\cos\lambda x + B\sin\lambda x + C\cosh\lambda x + D\sinh\lambda x \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)}$$

where $\lambda^4 = \frac{\rho A \omega^2}{EI}$ and the values of A, B, C and D depend on the end conditions of the bar.

So eqn. (8) is sued for free vibration Transverse.

Ex:4.4 free/free (free vibration)

From table 3 get the boundary conditions.

For a free end S = 0 and M = 0 (ie. no excitation)

From equation (4)

$$M = EI\frac{\partial^2 v}{\partial x^2} \quad \text{so that} \quad \frac{\partial^2 v}{\partial x^2} = 0 \ \text{and hence} \ \frac{\partial^2 V(x)}{\partial x^2} = 0$$

and from equation (5)

$$S = -EI\frac{\partial^3 v}{\partial x^3} \quad \text{so that} \quad \frac{\partial^3 v}{\partial x^3} = 0 \ \text{and hence} \ \frac{\partial^3 V(x)}{\partial x^3} = 0$$

Natural frequencies of a free/free bar (ie. no excitation at either end)

equation (8) gives

$$V(x) = A\cos\lambda x + B\sin\lambda x + C\cosh\lambda x + D\sinh\lambda x$$

differentiating with respect to x

$$\frac{\partial V(x)}{\partial x} = -A\lambda\sin\lambda x + B\lambda\cos\lambda x + C\lambda\sinh\lambda x + D\lambda\cosh\lambda x$$

and again

$$\frac{\partial^2 V(x)}{\partial x^2} = -A\lambda^2\cos\lambda x - B\lambda^2\sin\lambda x + C\lambda^2\cosh\lambda x + D\lambda^2\sinh\lambda x$$

and again

$$\frac{\partial^3 V(x)}{\partial x^3} = A\lambda^3\sin\lambda x - B\lambda^3\cos\lambda x + C\lambda^3\sinh\lambda x + D\lambda^3\cosh\lambda x$$

x = 0 is a free end so $\frac{\partial^2 V(0)}{\partial x^2} = 0$ and $\frac{\partial^3 V(0)}{\partial x^3} = 0$ which gives

$$\frac{\partial^2 V(0)}{\partial x^2} = -A + C = 0 \quad \ldots\ldots\ldots\ldots \quad \text{(a)}$$

$$\frac{\partial^3 V(0)}{\partial x^3} = -B + D = 0 \quad \ldots\ldots\ldots\ldots \quad \text{(b)}$$

x = L is also a free end so $\frac{\partial^2 V(L)}{\partial x^2} = 0$ and $\frac{\partial^3 V(L)}{\partial x^3} = 0$ which gives

$$\frac{\partial^2 V(L)}{\partial x^2} = -A\lambda^2\cos\lambda L - B\lambda^2\sin\lambda L + C\lambda^2\cosh\lambda L + D\lambda^2\sinh\lambda L = 0 \quad \ldots\ldots\ldots \quad \text{(c)}$$

$$\frac{\partial^3 V(L)}{\partial x^3} = A\lambda^3\sin\lambda L - B\lambda^3\cos\lambda L + C\lambda^3\sinh\lambda L + D\lambda^3\cosh\lambda L = 0 \quad \ldots\ldots\ldots \quad \text{(d)}$$

As there is no excitation the four equations (a), (b), (c) and (d) will give non-zero values for A, B, C and D only at natural frequencies.

From (a) C = A and from (b) D = B so substituting in (c) and (d)

$$-A\lambda^2\cos\lambda L - B\lambda^2\sin\lambda L + A\lambda^2\cosh\lambda L + B\lambda^2\sinh\lambda L = 0$$

$$A\lambda^3\sin\lambda L - B\lambda^3\cos\lambda L + A\lambda^3\sinh\lambda L + B\lambda^3\cosh\lambda L = 0$$

and rearranging

$$A(\cos\lambda L - \cosh\lambda L) + B(\sin\lambda L - \sinh\lambda L) = 0 \quad \ldots\ldots\ldots\ldots \quad \text{(e)}$$

$$A(\sin\lambda L + \sinh\lambda L) - B(\cos\lambda L - \cosh\lambda L) = 0 \quad \ldots\ldots\ldots\ldots \quad \text{(f)}$$

from (e)

$$B = -\frac{A(\cos\lambda L - \cosh\lambda L)}{(\sin\lambda L - \sinh\lambda L)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (g)$$

and substituting in (f)

$$A(\sin\lambda L + \sinh\lambda L) + \frac{A(\cos\lambda L - \cosh\lambda L)}{(\sin\lambda L - \sinh\lambda L)}(\cos\lambda L - \cosh\lambda L) = 0$$

$$\therefore A\left[\frac{\sin^2\lambda L - \sinh^2\lambda L + (\cos\lambda L - \cosh\lambda L)^2}{(\sin\lambda L - \sinh\lambda L)}\right] = 0$$

$$\therefore A\left[\frac{\sin^2\lambda L - \sinh^2\lambda L + \cos^2\lambda L - 2\cos\lambda L\cosh\lambda L + \cosh^2\lambda L}{(\sin\lambda L - \sinh\lambda L)}\right] = 0$$

$$\therefore A\left[\frac{2(1 - \cos\lambda L\cosh\lambda L)}{(\sin\lambda L - \sinh\lambda L)}\right] = 0$$

Thus A = 0 is a solution and there is no motion which is not unexpected as there is no excitation or when

$$1 - \cos\lambda L\cosh\lambda L = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

or A may have any value and vibration occurs, Thus equation (9) is the natural frequency equation for a free/free bar. As expected there is an infinite set of solutions since the system has an infinite number of degrees-of-freedom. The lower value solutions are,

From table 4 with respect to eqn. (9).

$\lambda_0 L = 0.0$; $\lambda_1 L = 4.73$; $\lambda_2 L = 7.853$; $\lambda_3 L = 10.996$; $\lambda_4 L = 14.137$; $\lambda_5 L = 17.279$; $\lambda_6 L = 20.42$;

since $\lambda^4 = \frac{\rho A\omega^2}{EI}$ rearranging gives $\omega = \frac{(\lambda L)^2}{L^2}\sqrt{\frac{EI}{\rho A}}$. Thus the lower natural frequencies are,

$\omega_{n0} = 0.0$; $\omega_{n1} = \frac{22.37}{L^2}\sqrt{\frac{EI}{\rho A}}$; $\omega_{n2} = \frac{61.67}{L^2}\sqrt{\frac{EI}{\rho A}}$; $\omega_{n3} = \frac{120.9}{L^2}\sqrt{\frac{EI}{\rho A}}$; $\omega_{n4} = \frac{199.9}{L^2}\sqrt{\frac{EI}{\rho A}}$; $\omega_{n5} = \frac{298.6}{L^2}\sqrt{\frac{EI}{\rho A}}$; $\omega_{n6} = \frac{417}{L^2}\sqrt{\frac{EI}{\rho A}}$

The zero frequency mode is expected because the system is free/free.

Mode shapes of a free/free bar

Each natural frequency has an associated mode shape. These are found from equation (8)

$$V(x) = A\cos\lambda x + B\sin\lambda x + C\cosh\lambda x + D\sinh\lambda x$$

with $C = A$ and $D = B$ and $B = -\dfrac{A(\cos\lambda L - \cosh\lambda L)}{(\sin\lambda L - \sinh\lambda L)}$ so that

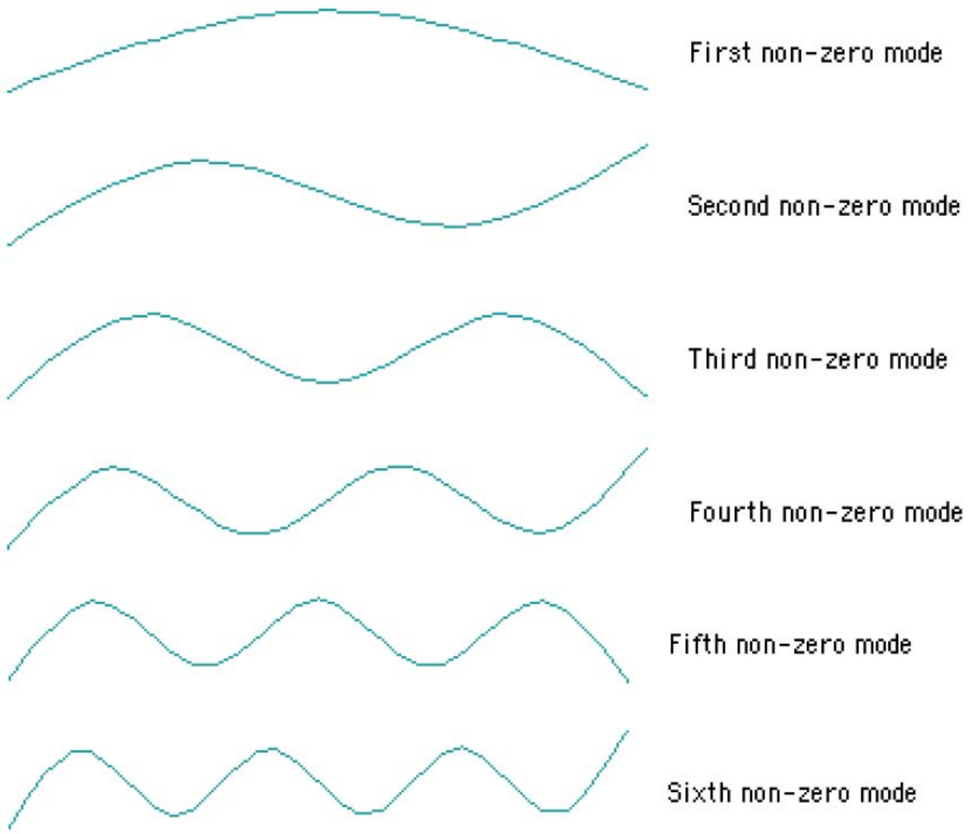
$$V(x) = A(\cos\lambda x + \cosh\lambda x) + B(\sin\lambda x + \sinh\lambda x)$$

and

$$\therefore V(x) = A\left[\frac{(\sin\lambda L - \sinh\lambda L)(\cos\lambda x + \cosh\lambda x) - (\cos\lambda L - \cosh\lambda L)(\sin\lambda x + \sinh\lambda x)}{(\sin\lambda L - \sinh\lambda L)}\right]$$

since it is the "shape" we are interested in then the value of $\dfrac{A}{(\sin\lambda L - \sinh\lambda L)}$ may be any value and the mode shape is given by,

$$V(x) = K\left[(\sin\lambda L - \sinh\lambda L)(\cos\lambda x + \cosh\lambda x) - (\cos\lambda L - \cosh\lambda L)(\sin\lambda x + \sinh\lambda x)\right]$$



Mode shapes of free/free bar

Ex:4.5 Clamped/free bar (free vibration)

Natural frequencies and modes of a clamped/free bar

If a similar analysis is completed for a clamped/free bar with the clamped end having no deflection and no slope it is found that the natural frequencies are given by,

$$1+\cos\lambda L\cosh\lambda L = 0 \quad \text{.......................}(10)$$

From table 4 .with respect to eqn. (10).

Again as expected there is an infinite set of solutions since the system has an infinite number of degrees-of-freedom. The lower value solutions are,

$$\lambda_1 L = 1.875;\ \lambda_2 L = 4.694;\ \lambda_3 L = 7.854;\ \lambda_4 L = 11.0;\ \lambda_5 L = 14.14;\ \lambda_6 L = 17.28;$$

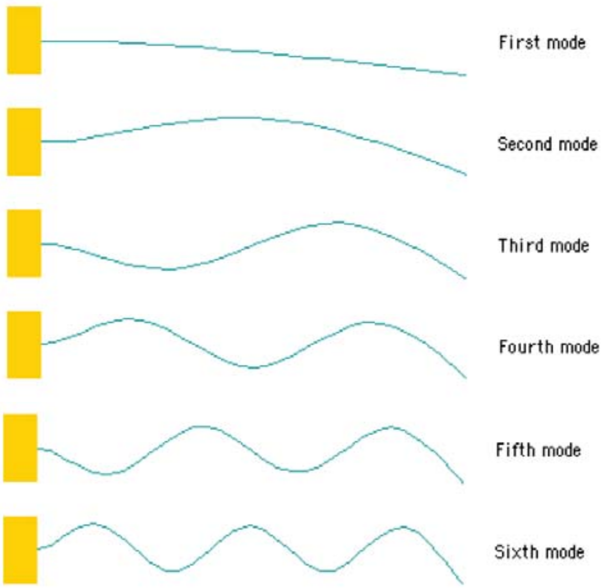
since $\lambda^4 = \frac{\rho A\omega^2}{EI}$ rearranging gives $\omega = \frac{(\lambda L)^2}{L^2}\sqrt{\frac{EI}{\rho A}}$. Thus the lower natural frequencies are,

$$\omega_{n1} = \frac{3.52}{L^2}\sqrt{\frac{EI}{\rho A}};\ \omega_{n2} = \frac{22.03}{L^2}\sqrt{\frac{EI}{\rho A}};\ \omega_{n3} = \frac{61.7}{L^2}\sqrt{\frac{EI}{\rho A}};\ \omega_{n4} = \frac{120.9}{L^2}\sqrt{\frac{EI}{\rho A}};$$

$$\omega_{n5} = \frac{199.9}{L^2}\sqrt{\frac{EI}{\rho A}};\ \omega_{n6} = \frac{298.6}{L^2}\sqrt{\frac{EI}{\rho A}}$$

Each natural frequency has an associated mode shape given by,

$$V(x) = K\left[(\sin\lambda L + \sinh\lambda L)(\cos\lambda x - \cosh\lambda x) - (\cos\lambda L + \cosh\lambda L)(\sin\lambda x - \sinh\lambda x)\right] \quad \text{.......} \quad (11)$$



Mode shapes of a clamped/free bar