End condition	Frequency Equation	$\lambda_1 L$	$\lambda_2 L$	$\lambda_3 L$	$\lambda_4 L$
Pinned- Pinned	**$\sin\lambda_n L=0$**	**π**	**2π**	**3π**	**4π**
Free- free	**$\cos\lambda_n L \cosh\lambda_n L=1$**	**4.730041**	**7.853205**	**10.995608**	**14.13716**
Fixed-fixed	**$\cos\lambda_n L \cosh\lambda_n L=1$**	**4.730041**	**7.853205**	**10.995608**	**14.13716**
Fixed-free	**$\cos\lambda_n L \cosh\lambda_n L=-1$**	**1.875104**	**4.691091**	**7.854757**	**10.99550**
Fixed=pinned	**$\tan\lambda_n L-\tanh\lambda_n L=0$**	**3.926602**	**7.068583**	**10.210176**	**13.35196**
Pinned-free	**$\tan\lambda_n L-\tanh\lambda_n L=0$**	**3.926602**	**7.068583**	**10.210176**	**13.35196**

($\lambda_0 L = 0$ for rigid body mode)

Table (4)Transverse Vibration

<u>FROCED Vibration.</u>

<u>Ex:4.6 free/free bar</u>

It is informative and useful to obtain the responses of a bar when excited at one end. Consider a free/free bar with a sinusoidal force applied at one end.

For an exciting force at one end. eg $S = F_L e^{i\omega t}$ at $x = L$

If there is only an exciting force at the end then there will be no moment and so $\frac{\partial^2 V(x)}{\partial x^2} = 0$ as for a free end.

From figure 1 it should be noted that at the right hand end when $x = L$ (the length of the bar) the force S is in the same positive direction as v.

Thus for $x = L$ when $S = F_L e^{i\omega t}$ from (5) $S = -EI\frac{\partial^3 v}{\partial x^3}$ so that $F_L e^{i\omega t} = -EI\frac{\partial^3 v}{\partial x^3}$

substituting for $v(x,t) = V(x)e^{i\omega t}$ gives $F_L = -EI\frac{\partial^3 V(x)}{\partial x^3}$ and hence $\frac{\partial^3 V(x)}{\partial x^3} = -\frac{F_L}{EI}$

Eqn. (8) gives.

When the excitation is at x = L then x = 0 is a free end so $\frac{\partial^2 V(0)}{\partial x^2} = 0$ and $\frac{\partial^3 V(0)}{\partial x^3} = 0$ which gives

$$\frac{\partial^2 V(0)}{\partial x^2} = -A + C = 0 \quad \ldots\ldots\ldots\ldots \quad (a)$$

$$\frac{\partial^3 V(0)}{\partial x^3} = -B + D = 0 \quad \ldots\ldots\ldots\ldots \quad (b)$$

At x = L is a forced end so $\frac{\partial^2 V(L)}{\partial x^2} = 0$ and $\frac{\partial^3 V(L)}{\partial x^3} = -\frac{F_L}{EI}$ which gives

$$\frac{\partial^2 V(L)}{\partial x^2} = -A\cos\lambda L - B\sin\lambda L + C\cosh\lambda L + D\sinh\lambda L = 0 \quad \ldots\ldots\ldots \quad (h)$$

$$\frac{\partial^3 V(L)}{\partial x^3} = A\lambda^3\sin\lambda L - B\lambda^3\cos\lambda L + C\lambda^3\sinh\lambda L + D\lambda^3\cosh\lambda L = -\frac{F_L}{EI} \quad \ldots\ldots \quad (i)$$

The four equations (a), (b), (h) and (i) allow the constants A, B, C and D to be found.
From (a) C = A and from (b) D = B so substituting in (h) and (i)

And solved to find that:

$$\omega_n = \frac{\pi^2}{\ell^2}\sqrt{\frac{EI}{\rho A}}\,(n^4 + \frac{n^2 F \ell^2}{\pi^2\, EI})^{1/2}$$

Deflected shape of free/free bar excited at one end.

5.Computational Methods

5.1**<u>Iteration method</u>**.

Matrix iteration is an approach for finding the Eigen values and Eigen vectors of a multi-DOF system depends on the subspace iteration method. The method is applicable to the equations of motion formulated by either the flexibility or by the stiffness matrices.

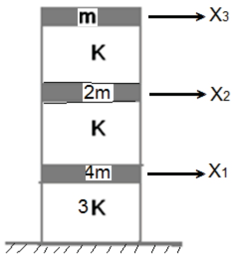
The equation for normal mode vibration is

$$[D]\{x\} = \lambda\{x\} \quad \dots\dots\dots(1) \qquad where\ \lambda = \frac{1}{\omega^2}$$

$$\{x\} \quad is\ trial\ vector\ , and\ [D] = [k]^{-1}[m] \qquad ..dynamic\ \ matrix$$

5.1 Example:

For the system shown in the figure , write the matrix equation based on flexibility and determine the lowest (first) natural frequency using iteration method.



Solution:

Since [a]= $[k]^{-1}$ by setting x_1=1 and x_2=x_3=0 then x_2=1 and x_1=x_3=0 ,and x_3=1 and x_1=x_2=0 the stiffness matrix is obtained as:

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} = k \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\therefore [a] = [k]^{-1} = \frac{1}{3k}\begin{bmatrix}1 & 1 & 1\\ 1 & 4 & 4\\ 0 & 4 & 7\end{bmatrix}, \; [m] = m\begin{bmatrix}4 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

From equation (1) $\quad [D]\{x\} = \lambda\{x\}$

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 4 & 4\\ 0 & 4 & 7\end{bmatrix}\begin{bmatrix}4 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix} = \frac{3k}{m\omega^2}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix}$$

So

$$\begin{bmatrix}4 & 2 & 1\\ 4 & 8 & 4\\ 4 & 8 & 7\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix} = \frac{3k}{m\omega^2}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix}$$ Note that $[D]$ is not symmetry

After preparing the above equation the following procedure to be follows.

1. Assume a set of amplitudes for the left column.
   Let it to be
   $$\{x\} = \begin{Bmatrix}0.2\\ 0.6\\ 1.0\end{Bmatrix}$$
   $$\therefore \begin{bmatrix}4 & 2 & 1\\ 4 & 8 & 4\\ 4 & 8 & 7\end{bmatrix}\begin{Bmatrix}0.2\\ 0.6\\ 1.0\end{Bmatrix} = \begin{Bmatrix}3.0\\ 9.6\\ 12.6\end{Bmatrix}$$
2. Normalize the new column by making one of the amplitudes equal to unity (Dividing each term of column by 12.6).
   $$\begin{Bmatrix}3.0\\ 9.6\\ 12.6\end{Bmatrix} = 12.6\begin{Bmatrix}0.238\\ 0.762\\ 1.000\end{Bmatrix}$$
3. The procedure is repeated with the normalized column until the amplitudes stabilize to a definite pattern.
   $$\begin{bmatrix}4 & 2 & 1\\ 4 & 8 & 4\\ 4 & 8 & 7\end{bmatrix}\begin{Bmatrix}0.238\\ 0.762\\ 1.000\end{Bmatrix} = \begin{Bmatrix}3.476\\ 11.048\\ 14.048\end{Bmatrix} = 14.048\begin{Bmatrix}0.247\\ 0.786\\ 1.000\end{Bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} 0.247 \\ 0.786 \\ 1.000 \end{Bmatrix} = \begin{Bmatrix} 3.56 \\ 11.267 \\ 14.267 \end{Bmatrix} = 14.276 \begin{Bmatrix} 0.249 \\ 0.79 \\ 1.000 \end{Bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} 0.249 \\ 0.79 \\ 1.000 \end{Bmatrix} = \begin{Bmatrix} 3.576 \\ 11.316 \\ 14.316 \end{Bmatrix} = 14.316 \begin{Bmatrix} 0.249 \\ 0.79 \\ 1.000 \end{Bmatrix}$$

After some iteration

$\frac{3k}{m\omega^2} = 14.316$ or $\omega_{n1} = 0.457\sqrt{\frac{k}{m}}$ and the corresponding mode shape is

$$\phi_1 = \begin{Bmatrix} 0.249 \\ 0.79 \\ 1.000 \end{Bmatrix}$$

<u>Converge to higher modes</u>:

From the expansion theorem , let the assumed trial vector is expressed in terms of the normal modes ϕ_i as shown below.

$\{x\} = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + \ldots.. + \ldots.$ (1)

Where

c_i *are constants*

Now, pre-multiply equation (1) by $\phi_1^T[m]$ where ϕ_1 is the first normal mode.

Hence

$\phi_1^T[m]\{x\} = c_1\phi_1^T[m]\phi_1 + c_2\phi_1^T[m]\phi_2 + c_3\phi_1^T[m]\phi_3 + \ldots..$

Due to Orthogonality, all terms on the right hand side of this equation except the first term are zero.

$\Phi_1^T[m]\{x\} = c_1\Phi_1^T[m]\Phi_1$ …… (2)

To remove ϕ_1 from equation (1) let $c_1 = 0$ in this case

$\phi_1^T[m]\{x\} = 0$ this called the constraint equation

And for a system of 3DOF

$$\phi_1^T[m]\{x\} = \{x_1^1 \quad x_2^1 \quad x_3^1\}\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0$$

Or

$$c_1 = m_1 x_1^1 x_1 + m_2 x_2^1 x_2 + m_3 x_3^1 x_3 = \sum_{i=1}^{3} m_i\, x_i^1 x_i = 0 \ldots\ldots\ldots(3)$$

From equation (3)

$$x_1 = -\frac{m_2}{m_1}\left(\frac{x_2}{x_1}\right)^1 x_2 - \frac{m_3}{m_1}\left(\frac{x_3}{x_1}\right)^1 x_3 \qquad \ldots\ldots.(a)$$

$$x_2 = x_2 \qquad \ldots\ldots..(b)$$

$$x_3 = x_3 \qquad \ldots\ldots..(c)$$

Or as a matrix form

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 0 & -\frac{m_2}{m_1}\left(\frac{x_2}{x_1}\right)^1 & -\frac{m_3}{m_1}\left(\frac{x_3}{x_1}\right)^1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = [s]\{x\}$$

Where $[s] = \mathit{sweeping\ matrix\ which\ is\ used\ for\ removing}\ \phi_1$

Now , by replacing $\{x\}$ on the left hand side of equation $[D]\{x\} = \lambda\{x\}$ becomes to

$$[D][s]\{x\} = \lambda\{x\}$$

Iteration of this equation now sweeping out the undesired ϕ_1 component and converges to the second mode Φ_2

5.2Example:

Find the second and third mode of vibration for the previous example. Where $\lambda_1 = 14.32$, and

$$\phi_1 = \begin{Bmatrix} 0.25 \\ 0.79 \\ 1.000 \end{Bmatrix}, [m] = m \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore [s] = \begin{bmatrix} 0 & -\frac{m_2}{m_1}(\frac{x_2}{x_1})^1 & -\frac{m_3}{m_1}(\frac{x_3}{x_1})^1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{2m}{4m}(\frac{0.79}{0.25})^1 & -\frac{m}{4m}(\frac{1}{0.25})^1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1.58 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And

$$[D][s] = \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 0 & -1.58 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.67 & 0 \\ 0 & 1.67 & 3 \end{bmatrix}$$

$$[D][s]\{x\} = \lambda\{x\}$$

$$\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.67 & 0 \\ 0 & 1.67 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \lambda \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Now , we start the same iteration procedure to find the second mode

So

Assume a set of amplitudes for left column

Let $\{x\} = \begin{Bmatrix} 0.5 \\ -0.2 \\ 1.0 \end{Bmatrix}$

The first iteration then becomes

$$\begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.67 & 0 \\ 0 & 1.67 & 3 \end{bmatrix} \begin{Bmatrix} 0.5 \\ -0.2 \\ 1.0 \end{Bmatrix} = \begin{Bmatrix} -2.136 \\ -0.334 \\ 2.66 \end{Bmatrix} = 2.66 \begin{Bmatrix} -0.801 \\ -0.125 \\ 1.00 \end{Bmatrix}$$

Then after a few more iteration it converge to

$$3.0 \begin{Bmatrix} -1.00 \\ 0 \\ 1.00 \end{Bmatrix} = \frac{3k}{m\omega^2} \begin{Bmatrix} -1.00 \\ 0 \\ 1.00 \end{Bmatrix} \quad \text{so } \omega_{n2} = \sqrt{\frac{k}{m}} \text{ and } \phi_2 = \begin{Bmatrix} -1.00 \\ 0 \\ 1.00 \end{Bmatrix}$$

For the determination of the third mode, we impose the condition $c_1 = c_2 = 0$

The general form of equation (3)

$$C_j = \sum_{i=1}^{3} m_i \, x_i^1 x_i = 0 \qquad for \qquad c_1 \text{ and } c_2$$

$$C_1 = \sum_{i=1}^{3} m_i \, x_i^1 x_i = \quad m_1 x_1^1 x_1 + m_2 x_2^1 x_2 + m_3 x_3^1 x_3$$

$$= 4 * 0.25 * x_1 + 2 * 0.79 * x_2 + 1 * 1.0 * x_3$$

$$C_2 = \sum_{i=1}^{3} m_i \, x_i^2 x_i = \quad m_1 x_1^{(2)} x_1 + m_2 x_2^{(2)} x_2 + m_3 x_3^{(2)} x_3$$

$$= 4 * (-1.0) * x_1 + 2 * (0) * x_2 + 1 * 1.0 * x_3$$

From these two equations we obtain

$x_1 = 0.25\, x_3$, $x_2 = -0.79\, x_3$, $x_3 = x_3$

Or as a matrix form

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & -0.79 \\ 0 & 0 & 1.00 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$[D][s]\{x\} = \lambda\{x\}$$

$$\begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & -0.79 \\ 0 & 0 & 1.00 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{3k}{m\omega^2} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Now, for the above values of $x_1 = 0.25$, $x_2 = -0.79$, $x_3 = 1.00$

$$\begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & -0.79 \\ 0 & 0 & 1.00 \end{bmatrix}\begin{Bmatrix} 0.25 \\ -0.79 \\ 1.00 \end{Bmatrix} = 1.68 = \frac{3k}{m\omega^2}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

So

$$\omega_{n3} = 1.34\sqrt{\frac{k}{m}} \text{ and } \phi_3 = \begin{Bmatrix} 0.25 \\ -0.79 \\ 1.00 \end{Bmatrix}$$

5.2 Transformation of coordinates

The matrix equation for the normal mode of free un damped vibration is generally written as:

$$\left[-\omega^2[m] + [k]\right]\{x\} = \{0\} \ldots\ldots\ldots\ldots(1)$$

Pre-multiplying equation (1) by $[m]^{-1}$

$$\left[-\omega^2[I] + [m]^{-1}[k]\right]\{x\} = \{0\}$$

Or

$$\left[-\bar{\lambda}[I] + [\bar{D}]\right]\{x\} = \{0\} \ \ldots\ldots\ldots(2)$$

Where $\bar{\lambda} = \omega^2$, $[\bar{D}] = [m]^{-1}[k]$ = dynamic matrix

Now, Pre-multiplying equation (1) by $[k]^{-1}$

$$\left[[D] - \lambda[I]\right]\{x\} = \{0\} \ldots\ldots\ldots\ldots\ldots(3) \qquad \lambda = \frac{1}{\omega^2}$$

Where $[\bar{D}]$ and $[D]$ are different , but both called dynamic matrix., generally $[\bar{D}]$ and $[D]$ are not symmetric, so to obtain the standard form of the equation of motion we introduce the following transformation of coordinates .

$$\{x\} = [U]^{-1}\{Y\}$$

Where U is called upper matrix

$\therefore$ $equation$ (1) $can\ be\ written\ as$:

$$\left[-\bar{\lambda}[m][U]^{-1} + [k][U]^{-1}\right]\{Y\} = \{0\}$$

Pre-multiplying this equation by $[U]^{-T}$

$$\left[-\bar{\lambda}\,[U]^{-T}\,[m][U]^{-1} + \,[U]^{-T}[k][U]^{-1}\right]\{Y\} = \{0\} \ldots\ldots\ldots\ldots..(4)$$

Now,

1. decompose $[m] = [U]^T[U]$

$\therefore$ *equation* (4) *become*

$$\left[-\bar{\lambda}[I] + \,[U]^{-T}[k][U]^{-1}\right]\{Y\} = \{0\}$$

2. decompose $[k] = [U]^T[U]$

$\therefore$ *equation* (4) *become*

$$\left[\,[U]^{-T}\,[m][U]^{-1} + \,\lambda[I]\right]\{Y\} = \{0\}$$

Both equations are in the standard Eigen value form.

Note:

If the mass matrix is diagonal (lumped –mass)

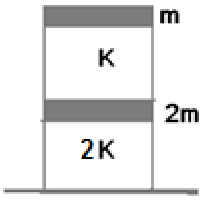
$$[m] = \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{bmatrix} \text{ then } [U] = \begin{bmatrix} \sqrt{m_{11}} & 0 & 0 \\ 0 & \sqrt{m_{22}} & 0 \\ 0 & 0 & \sqrt{m_{33}} \end{bmatrix}$$

And

$$[U]^{-1} = [U]^{-T} = \begin{bmatrix} 1/\sqrt{m_{11}} & 0 & 0 \\ 0 & 1/\sqrt{m_{22}} & 0 \\ 0 & 0 & 1/\sqrt{m_{33}} \end{bmatrix}$$

5.3Example:

For a 2DOF system if $[m] = m\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}$,and $[k] = k\begin{bmatrix}3 & -1\\-1 & 1\end{bmatrix}$ find the natural frequencies and normal modes using the decomposing of mass matrix.



Solution:

Since the mass matrix is for lamped mass.

$$\therefore\ [U] = [M]^{1/2} = \sqrt{m}\begin{bmatrix}\sqrt{2} & 0\\ 0 & 1\end{bmatrix} \text{ and } [U]^{-1} = [U]^{-T} = \frac{1}{\sqrt{m}}\begin{bmatrix}1/\sqrt{2} & 0\\ 0 & 1\end{bmatrix}$$

$$\therefore\ [U]^{-T}\,[m][U]^{-1} = [U]^{-T}[U]^{T}[U][U]^{-1} = [I]$$

so

$$\left[-\bar{\lambda}[I] + [U]^{-T}[k][U]^{-1}\right]\begin{Bmatrix}Y_1\\Y_2\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

$$= \left[-\omega^2[I] + \frac{k}{m}\begin{bmatrix}1/\sqrt{2} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 0\\ 0 & 1\end{bmatrix}\right]\begin{Bmatrix}Y_1\\Y_2\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

$$= \left[\begin{bmatrix}1.5 & -0.707\\-0.707 & 1\end{bmatrix} - \frac{m\omega^2}{k}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right]\begin{Bmatrix}Y_1\\Y_2\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

Now, let $\frac{m\omega^2}{k}$ =q

$$\begin{vmatrix}1.5-q & -0.707\\-0.707 & 1-q\end{vmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix} \quad \Rightarrow q^2\text{-}2.5\,q + 1 = 0$$

$$q_1 = 0.5 \Rightarrow \quad \omega_{n1} = 0.707\sqrt{\frac{k}{m}}\ ,\ \phi_1 = \begin{Bmatrix}Y_1\\Y_2\end{Bmatrix} = \begin{Bmatrix}0.707\\1.00\end{Bmatrix}$$

$$q_2 = 2.0 \Rightarrow \quad \omega_{n2} = 1.414\sqrt{\frac{k}{m}} \; , \phi_2 = \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} -1.414 \\ 1.00 \end{Bmatrix}$$

Now. We have the mode in the Y coordinate to change it to X coordinate

$$[Y] = \begin{bmatrix} 0.707 & -1.414 \\ 1.00 & 1.00 \end{bmatrix}$$

$$[X] = [U]^{-1}Y = \begin{bmatrix} 0.707 & -1.414 \\ 1.00 & 1.00 \end{bmatrix}\begin{bmatrix} 0.707 & 0 \\ 0 & 1.00 \end{bmatrix} = \begin{bmatrix} 0.5 & -1.00 \\ 1.00 & 1.00 \end{bmatrix}$$

5.3 <u>Cholesky Decomposition:</u>

When $[M]$ or $[K]$ is full matrices (i.e not a symmetric matrix) $[U]$ and $[U]^{-1}$ can be found from the Cholesky Decomposition as follows:

Write $[M] = [U]\,[U]^T$ or $[K] = [U]\,[U]^T$ as:

$$[U]^T \qquad [U] \qquad = \qquad [M]$$

$$\begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix}$$

Or

$$[U]^T \qquad [U] \qquad = \qquad [K]$$

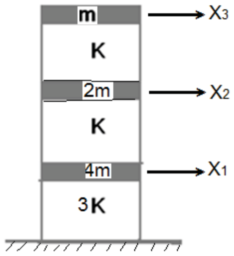
$$\begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix}$$

The inverse of the upper matrix is can be determined from

$$[U][U]^{-1} = [I]$$

5.4 Example:

A three –story model building shown in the figure below for which the equation of motion is



$$\left[-\frac{\omega^2 m}{k}\begin{bmatrix}4 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix}+\begin{bmatrix}4 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}\right]\begin{Bmatrix}x_1\\x_2\\x_3\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

Reduce the equation to the standard form by decomposing the stiffness matrix according to the Cholesky Decomposition.

Solution:

Step 1

$$[U]^T \qquad [U] \qquad = \qquad [K]$$

$$\begin{bmatrix}u_{11} & 0 & 0\\u_{12} & u_{22} & 0\\u_{13} & u_{23} & u_{33}\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\0 & u_{22} & u_{23}\\0 & 0 & u_{33}\end{bmatrix}=\begin{bmatrix}k_{11} & k_{12} & k_{13}\\k_{12} & k_{22} & k_{23}\\k_{13} & k_{23} & k_{33}\end{bmatrix}=\begin{bmatrix}4 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}$$

$$\begin{bmatrix}u_{11}^2 & u_{11}u_{12} & u_{11}u_{13}\\u_{11}u_{12} & u_{12}^2+u_{22}^2 & u_{12}u_{13}+u_{22}u_{23}\\u_{11}u_{13} & u_{12}u_{13}+u_{22}u_{23} & u_{13}^2+u_{23}^2+u_{33}^2\end{bmatrix}=\begin{bmatrix}4 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}$$

Hence

$$[U]=\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}=\begin{bmatrix} 2 & -0.5 & 0 \\ 0 & 1.3228 & -0.7229 \\ 0 & 0 & 0.6547 \end{bmatrix}$$

Check by substituting back into $[U]^T\,[U] = [K]$

Step 2:

Find the inverse of $[U]$ from

$$[U][U]^{-1}=[I]$$

$$\begin{bmatrix} 2 & -0.5 & 0 \\ 0 & 1.3228 & -0.7229 \\ 0 & 0 & 0.6547 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[U]^{-1}=[b_{ij}]=\begin{bmatrix} 0.5 & 0.1889 & 0.2182 \\ 0 & 0.7559 & 0.8726 \\ 0 & 0 & 1.5275 \end{bmatrix}$$

Check by substituting back into $[U][U]^{-1}=[I]$

Step 3:

A cording using the decomposing the stiffness matrix [K]

$$\left[\,[U]^{-T}\,[m][U]^{-1}+\lambda[I]\right]\{Y\}=\{0\}$$

so

$[U]^{-T}\,[m][U]^{-1}=$

$$\begin{bmatrix} 0.5 & 0 & 0 \\ 0.1889 & 0.7559 & 0 \\ 0.2182 & 0.8726 & 1.5275 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.5 & 0.1889 & 0.2182 \\ 0 & 0.7559 & 0.8726 \\ 0 & 0 & 1.5275 \end{bmatrix}$$

$$=\begin{bmatrix} 1.00 & 0.3779 & 0.436 \\ 0.3779 & 0.7559 & 1.484 \\ 0.436 & 1.4840 & 1.5275 \end{bmatrix}$$ Note that is a symmetric

Step 4:

The equation of motion is now in standard form but in Y coordinate

$$\left[\begin{bmatrix} 1.00 & 0.3779 & 0.436 \\ 0.3779 & 0.7559 & 1.484 \\ 0.43 & 1.4840 & 1.5275 \end{bmatrix} - \frac{k}{\omega^2 m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\begin{Bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

*5.4*Jacobi Diagonalization method:

From previous chapter we know that the assembling of the orthonormal eigenvector $\{\tilde{\phi}\}$ in to the modal matrix $[\tilde{P}]$ gives:

$$[\tilde{P}]^T[M][\tilde{P}] = [I]$$

And $[\tilde{P}]^T[K][\tilde{P}] = [\Lambda] = \begin{bmatrix} \ddots & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \ddots \end{bmatrix}$

The Jacobi method is based on the principle that any real symmetric matrix $[A]$ has only real Eigen values can be diagonalized. The method is developed for the standard Eigen problem equation:

$$[[A] - \lambda[I]]\{Y\} = \{0\}$$

The kth is iteration step is defined by the following equation .

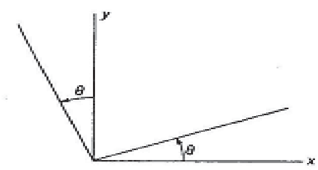
$[R]_k^T[A][R]_k = [A]_k$ k=1

Then

$[R]_{k+1}^T[A]_k[R]_{k+1} = [A]_{k+1}$ k=1,2 , etc

Where

$[R]$ *is rotation matrix* , for 2x2 matrix $=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



Notes:

1. *The rotation matrix [R] used to rotate the axes through an angle θ.*
2. *Since* $[R]^T[R] = [R][R]^T = [I] \Rightarrow$ *the matrix* $[R]$ *is orthonormal.*
3. *If* [A]= $\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$=*2x2 matrix , i.e. there is only one off-diagonal element* a_{12}*, and the Eigen problem is solved in a single step.*

Or

$$[R]^T[\mathrm{A}]\,[R] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

So,

$$\lambda_1 = a_{11}\cos^2\theta + 2\,a_{12}\sin\theta\cos\theta + a_{22}\sin^2\theta \quad \dots\dots\dots (a)$$

$$\lambda_2 = a_{11}\sin^2\theta - 2\,a_{12}\sin\theta\cos\theta + a_{22}\cos^2\theta \quad \dots\dots\dots (b)$$

$$0 = -(a_{11} - a_{22})\sin\theta\cos\theta + a_{12}(\cos^2\theta - \sin^2\theta) \quad \dots\text{(c)}$$

From equation (c)

Since $\sin\theta\cos\theta = \sin 2\theta$, and $\cos^2\theta - \sin^2\theta = \frac{\cos 2\theta}{2}$

Then

$$\tan 2\theta = \frac{2a_{12}}{a_{11} - a_{22}}$$

4.The Eigenvector corresponding to the two Eigen values are represented by the two column of the rotating matrix $[R]$ *which is equal to* $[\tilde{P}]$.

Generally for n[th] order matrix , the rotation matrix $[R]$ is a unit matrix $[I]$ with the rotation matrix superimposed to align with the (i,j) off-diagonal element to be zeroed . For example to eliminate the element $a_{3,5}$ in a 6x6 matrix , the rotation matrix is:

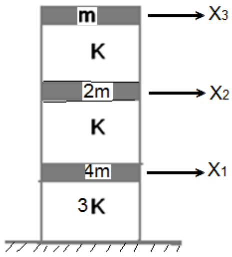
$$[R] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Where

$$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}} = \frac{2a_{35}}{a_{33} - a_{55}} \qquad \text{, i=3 , j=5}$$

Example 5.5

The system shown in the figure below which a 3DOF mode building for



which the equation of motion is.

$$\left[-\frac{\omega^2 m}{k}\begin{bmatrix}4 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix}+\begin{bmatrix}4 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}\right]\begin{Bmatrix}x_1\\x_2\\x_3\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

Use the Jacobi method to find its Eigen values and Eigen vectors .

Solution:

This equation is non standard Eigen problem, so to reduced it to the standard form, using the decomposing of mass matrix.

Since

$$[M]=\begin{bmatrix}4 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix}\Rightarrow [U]=\begin{bmatrix}\sqrt{4} & 0 & 0\\0 & \sqrt{2} & 0\\0 & 0 & \sqrt{1}\end{bmatrix}=\begin{bmatrix}2 & 0 & 0\\0 & 1.414 & 0\\0 & 0 & 1\end{bmatrix}$$

And

$$[U]^{-1}=[U]^{-T}=\begin{bmatrix}1/\sqrt{m_{11}} & 0 & 0\\0 & 1/\sqrt{m_{22}} & 0\\0 & 0 & 1/\sqrt{m_{33}}\end{bmatrix}=\begin{bmatrix}0.5 & 0 & 0\\0 & 0.707 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\therefore [U]^{-T}[k][U]^{-1}=\begin{bmatrix}1 & -0.3535 & 0\\-0.3535 & 1 & -0.707\\0 & -0.707 & 1\end{bmatrix}$$

So the standard form of the equation of motion becomes as:

$[-\lambda[I] + [A]\{Y\}\{0\}$ or

$$\left[-\lambda[I] + \begin{bmatrix} 1 & -0.3535 & 0 \\ -0.3535 & 1 & -0.707 \\ 0 & -0.707 & 1 \end{bmatrix}\right]\begin{Bmatrix} Y_1 \\ Y_2 \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$ But in Y coordinate

$where\ \lambda = \dfrac{\omega^2 m}{k}$

Now, the method start

<u>Step 1</u>.

First zero the largest off-diagonal term , which is $a_{23} = -0.707$

$$\therefore \tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}} = \frac{2a_{23}}{a_{22} - a_{33}} = \frac{2(-0.707)}{1-1} = \pm\infty$$

$2\theta = 90^0 \quad \Rightarrow \ \theta = 45^0$,cos 45=sin 45 $= 0.707$

$$\therefore [R]_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.707 & -0.707 \\ 0 & 0.707 & 0.707 \end{bmatrix}$$

$k = 1$

$$\therefore [A]_1 = [R]_1^T[\mathrm{A}][R]_1$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.707 & 0.707 \\ 0 & -0.707 & 0.707 \end{bmatrix}\begin{bmatrix} 1 & -0.3535 & 0 \\ -0.3535 & 1 & -0.707 \\ 0 & -0.707 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.707 & -0.707 \\ 0 & 0.707 & 0.707 \end{bmatrix}$$

$$\therefore [A]_1 = \begin{bmatrix} 1 & -0.25 & 0.25 \\ -0.25 & 0.2929 & 0 \\ 0.25 & 0 & 1.7071 \end{bmatrix}$$ note that $a_{13} = a_{31} \neq 0$

<u>Step 2</u>. Zero the term $a_{12} = -0.25$, $i = 1$, $j = 2$

$$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}} = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2(-0.25)}{1 - 0.2929} = -0.707$$

$$\theta = -17.63^0 \ \Rightarrow \ [R]_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.953 & 0.3029 & 0 \\ -0.3029 & 0.953 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$k = 1$

$$\therefore [A]_2 = [R]_2^T [\mathrm{A}]_1 [R]_2 = \begin{bmatrix} 1.097 & 0 & 0.2383 \\ 0 & 0.2134 & 0.0757 \\ 0 & 0.0757 & 1.7071 \end{bmatrix}$$

<u>Step3.</u>

To complete the first sweep of all the off-diagonal term zero the term a_{13}

$$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}} = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{2(0.2383.25)}{1.097 - 1.7071} = -0.7812$$

$$\theta = -18.998^0 \Rightarrow [R]_3 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 0.9455 & 0 & 0.3255 \\ 0 & 1 & 0 \\ -0.3255 & 0 & 0.9455 \end{bmatrix}$$

k=2

$$\therefore [A]_3 = [R]_3^T [\mathrm{A}]_2 [R]_3 = \begin{bmatrix} 1.0147 & -0.0240 & 0.0 \\ -0.0240 & 0.2134 & 0.071 \\ 0.0 & 0.071 & 1.817 \end{bmatrix}$$

$-0.0240, 0.071 \simeq 0$ so we can say the diagonal of the above matrix is represent λ_2, λ_1 $, and\ \lambda_3\ respectively\ (\lambda_1 < \lambda_2 < \lambda_3\ \ldots..etc)$

Where $\lambda_2 = 1.0147$, $\lambda_1 = 0.2134$, $and\ \lambda_3 = 1.817$

$$\omega_{n1} = 0.462\sqrt{\frac{k}{m}}\ , \omega_{n2} = \sqrt{\frac{k}{m}}\ , \omega_{n2} = 1.348\sqrt{\frac{k}{m}}$$

,,

$[\tilde{P}]_Y = [R]_1 [R]_2 [R]_3$

$$[\tilde{P}]_Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.707 & -0.707 \\ 0 & 0.707 & 0.707 \end{bmatrix} \begin{bmatrix} 0.953 & 0.3029 & 0 \\ -0.3029 & 0.953 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.9455 & 0 & 0.3255 \\ 0 & 1 & 0 \\ -0.3255 & 0 & 0.9455 \end{bmatrix}$$

$$[\tilde{P}]_Y = \begin{bmatrix} 0.9011 & 0.3029 & 0.3102 \\ 0.0276 & 0.6739 & -0.783 \\ -0.4327 & 0.6739 & 0.5988 \end{bmatrix}$$

$$\therefore\ [\tilde{P}]_X = [U]^{-1}[\tilde{P}]_Y = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.9011 & 0.3029 & 0.3102 \\ 0.0276 & 0.6739 & -0.783 \\ -0.4327 & 0.6739 & 0.5988 \end{bmatrix}$$

$$[\tilde{P}]_X = \begin{bmatrix} 0.4006 & 0.1515 & 0.1551 \\ 0.0195 & 0.4765 & -0.5221 \\ -0.4327 & 0.6739 & 0.5988 \end{bmatrix}$$

When normalized to 1.00, and from $[A]_3$

$$\begin{array}{ccc} \text{mode 2} & \text{mode 1} & \text{mode3} \\ \downarrow & \downarrow & \downarrow \end{array}$$

$$[\tilde{P}]_X = \begin{bmatrix} -0.925 & 0.225 & 0.259 \\ -0.045 & 0.707 & -0.872 \\ 1.00 & 1.00 & 1.00 \end{bmatrix}$$

6.Classical Methods:

The exact analysis for the vibrating systems of n DOF is generally difficult and its associated calculations are laborious and the results beyond the first few normal modes are often unreliable and meaningless. For this reasons Rayleigh's method and Dunkerley's equation are of great values and importance for the first modes and frequencies estimation.

6.1Rayleiy's method:

The method is based on Rayleigh's principle which can be stated as follows:

The frequency of vibration of conservation system vibrating about an equilibrium position has a stationary value in the neighborhood of nature mode. This stationary value in the fact is a minimum value in the neighborhood of the fundamental nature mode.

Now, let [M] and [K] be the mass and stiffness matrices respectively, then the kinetic and potential energy of an n-degree of freedom discrete system can be expressed as:

$$T_{max} = \frac{1}{2}\{\phi\}^T[M]\{\phi\}\omega^2$$

$$U_{max}=\frac{1}{2}\{\phi\}^T[K]\{\phi\}$$

By equating T_{max} and U_{max} it can obtained :

$$\omega^2 = R(\omega) = \frac{\{\phi\}^T[K]\{\phi\}}{\{\phi\}^T[M]\{\phi\}}$$

Where

$R(\omega) \dots\dots Rayleigh's\ quotiont.$

Example:

Estimate the fundamental frequency of vibration of the system shown , let $m_1= m_2= m_3=m$, $k_1= k_2=k_3 =k$, and the mode shape is

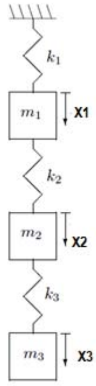
$$\{\phi\} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

Solution:

The stiffness and mass matrices are:

$$[K] = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[M] = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\omega^2 = \frac{\{\phi\}^T[K]\{\phi\}}{\{\phi\}^T[M]\{\phi\}} = \frac{(1\ 2\ 3)k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}}{(1\ 2\ 3)m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}}$$

$\omega = 0.4629\sqrt{\frac{k}{m}}$ And the exact solution from Example 3.1 is $\omega = 0.445\sqrt{\frac{k}{m}}$

$$\{\phi\}^{(1)} = \begin{Bmatrix} 1 \\ 1.8014 \\ 2.246 \end{Bmatrix}$$

<u>Field of application of Rayleigh's method:</u>

1. **Lumped masses**: The Rayleigh's method can be used to determine the fundamental frequency of a beam or shafts represented by a series of lumped masses . As a first approximation , a static deflection curve due to loads (i.e M_1g, M_2g ,…., and so on , and the corresponding deflections y_1,y_2 ,..., . The strain energy stored in the beam is determined from the work done by these loads , and the maximum potential and kinetic energies becomes.

$$U_{max} = \frac{1}{2}\ g(M_1y_1 + M_2y_2 + \cdots)$$

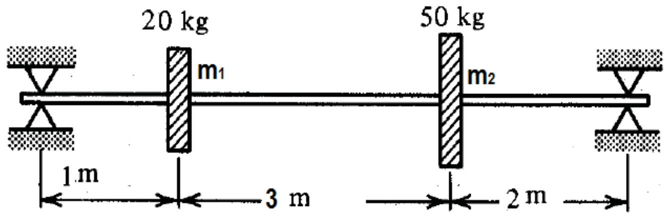
$$T_{max} = \frac{1}{2}\omega^2(M_1y_1^2 + M_2y_2^2 + \cdots)$$

By equating $T_{max}\ and\ U_{max}$ it can obtained :

$$\omega^2 = \frac{g\sum_i M_iy_i}{\sum_i M_i{y_i}^2}$$

Example:

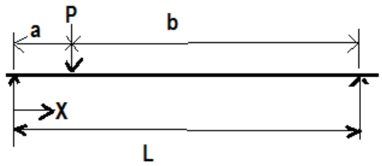
Estimate the fundamental frequency of lateral vibration of the system shown below:



Solution:

Let y(x) to be the deflection of the beam at any point x is:

$$y(x) = \begin{cases} \dfrac{pbx}{6EIl}(l^2 - b^2 - x^2) \quad \ldots.(1) \quad at\ 0 \le x \le a \\ -\dfrac{pa(l-x)}{6EIl}(a^2 + x^2 - 2lx) \quad \ldots.(2)\ \ at\ a \le x \le l \end{cases}$$



Let $\bar{y}_1$ be the deflection of m_1 due to m_1g, from equation (1)

x=1, *l=6, b=5, P=* m_1g

$\bar{y}_1 = 272.5/EI$

$\bar{\bar{y}}_1$ be the deflection of m_1 due to m_2g from equation (1)

x=1, *l=6, b=2, P=* m_2g

$\bar{\bar{y}}_1 = 844.75/EI$

And from equation (2) find.

$\bar{y}_2$, $\bar{\bar{y}}_2$

Hence

$\bar{y}_2 = 337.9/EI$ *The deflection of m_2* due to m_1g *at x=4 , a=1, l=6*

$\bar{\bar{y}}_2 = 1744/EI$ *The deflection of m_2* due to m_2g at a=2, *x=4, l=6*

The summation of m_1: $y_1 = \bar{y}_1 + \bar{\bar{y}}_1 = 1117.25/EI$

And of m_2 : $y_2 = \bar{y}_2 + \bar{\bar{y}}_2 = 2081.9/EI$

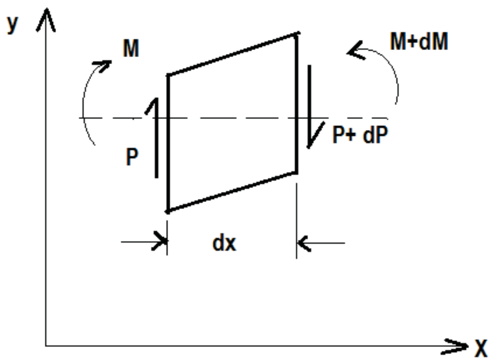
So

$$\omega^2 = \frac{g\sum_i M_i y_i}{\sum_i M_i {y_i}^2}$$

$\omega = [\frac{9.81(20*1117.25+50*2081.9)EI}{20*(1117.25)^2+50*(2081.9)^2}]^{1/2}$ = 0.07164 $\sqrt{EI}$ rad/s

2. **Continuous System:** For this application we have two methods to calculate the Rayleigh's.

A. **Differentiation method:**
In order to apply Raleigh's method we need to derive an expression for the maximum kinetic and potential energies.



$$T = \frac{1}{2}\int_0^l \dot{y}^2\, dm = \frac{1}{2}\int_0^l \dot{y}^2\, \rho A(x) dx$$

Assume a harmonic of y(x,t) as:
y(x,t)=Y(x)G(t) = Y(x)cos ωt ……(1)

$$\dot{y} = -\omega y \sin\omega t = -\omega Y(x) \Rightarrow \dot{y}^2 = \omega^2 Y_{(x)}^2$$

$$let \sin\omega t = 1$$

$$\boxed{\therefore\ T_{max} = \frac{\omega^2}{2}\int_0^l \rho A(x)\, Y_{(x)}^2 dx}$$

The potential energy of the beam can be found from the work done by the elastic force to the neutral configuration by disregarding the work done by the shear force.

$U = \frac{1}{2}\int_0^l M d\theta$……………..(2)

Since $\theta = \frac{dy}{dx}$, $\frac{1}{R} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$

R… radius of curvature of the neutral axis.

And from the theory of beam .

$\frac{1}{R} = \frac{M}{EI} \Rightarrow M = EI(\frac{d^2y}{dx^2})$

$\therefore U = \frac{1}{2}\int_0^l EI(\frac{d^2y}{dx^2})\,(\frac{d^2y}{dx^2})dx = \frac{1}{2}\int_0^l EI\,(\frac{d^2y}{dx^2})^2 dx$

From equation (1) the maximum value of y(x,t) is $Y_{(x)}$

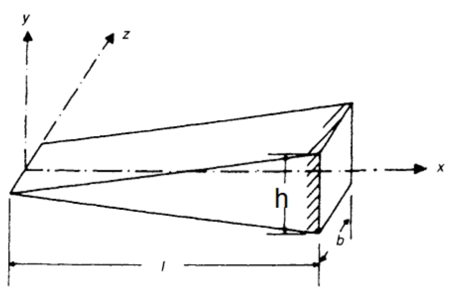
$$\boxed{\therefore U_{max} = \frac{1}{2}\int_0^l EI_{(x)}\,(\frac{d^2y}{dx^2})^2 dx}$$

By equating $T_{max}= U_{max}$ it can be obtained that:

$R(\omega) = \omega^2 = \frac{\int_0^l EI(\frac{d^2y}{dx^2})^2 dx}{\int_0^l \rho A(x) Y_{(x)}^2 dx}$ (a)

Example:

Find the fundamental frequency of transverse vibration of the non-uniform cantilever beam shown , using the deflection shape



$Y_{(x)} = (1 - \frac{x}{l})^2$, b=1

Solution:

Note: $Y_{(l)} = 0$, $Y_{(0)} = 1$

$A(x) = \dfrac{hx}{l}$

$I(x) = \dfrac{1}{12}\,(\dfrac{hx}{l})^3$

Or

$I(x) = \frac{bh^3}{12}$, h=$\frac{hx}{l}$, A=hb

Since

$Y_{(x)} = (1 - \frac{x}{l})^2 \;\Rightarrow \frac{d^2Y}{dx^2} = \frac{2}{l^2}$

$$\omega^2 = \frac{\int_0^l EI(x)\,(\frac{d^2y}{dx^2})^2 dx}{\int_0^l \rho A(x)\, Y_{(x)}^2 dx}$$

$$\omega^2 = \frac{\int_0^l E\frac{1}{12}\,(\frac{hx}{l})^3(\frac{2}{l^2})^2 dx}{\int_0^l \rho\frac{hx}{l}(1-\frac{x}{l})^4 dx} = 2.5\,\frac{Eh^2}{\rho l^4}$$

Or

$$\omega = 1.5811\sqrt{\frac{Eh^2}{\rho l^4}}$$

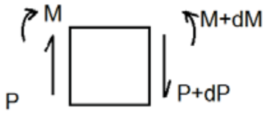
The exact value of ω is known to be

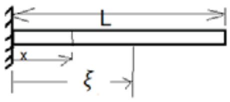
$$\omega = 1.5343\sqrt{\frac{Eh^2}{\rho l^4}}$$

B- Integration method:

In using Rayleigh's method of determine the fundamental frequency , we must choose an assumed curve, Although the deviation of this assumed deflection curve compared to the exact curve may be slight, its derivative could be in error by a large amount then $U = \frac{1}{2}\int_0^l EI_{(x)}\,(\frac{d^2y}{dx^2})^2 dx$ may be incorrect.

To avoid this difficulty, the following integration method for evaluating U is recommended for some beam problems





$U = \frac{1}{2}\int_0^l \frac{M_{(x)}^2}{EI} dx$ …….. (b)

Since

$P = \frac{dm}{dx}$

From F.B.D, and in the end of the beam.

$$P(\xi) = \omega^2 \int_{\xi}^{l} m(\xi)\, y(\xi) d\xi$$

M(x)= $\int_x^l P(\xi) d\xi$

x… displacement denoted the location of moment M

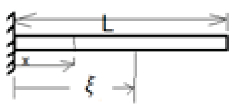
ξ…. displacement denoted the location of shear P

And,

T_{max}= $\frac{1}{2}\int_0^l \dot{y}^2\, \rho A(x) dx$

Example:

Determine the fundamental frequency of the uniform cantilever beam using simple curve $y = cx^2$ (the Exact solution is $= 3.52\sqrt{\frac{EI}{ml^4}}$)



Solution:

1- Using equation (a)

$$\omega^2 = \frac{\int_0^l EI(\frac{d^2y}{dx^2})^2 dx}{\int_0^l \rho A Y_{(x)}^2 dx} \quad , \frac{d^2y}{dx^2} = 2c$$

$$= \frac{EI\int_0^l 4c^2 dx}{\rho A \int_0^l c^2 x^4 dx} = \frac{EI4c^2 l}{\rho A c^2 \frac{l^5}{5}} = \frac{20EI}{\rho A l^4} \quad \Rightarrow \omega = 4.47\sqrt{\frac{EI}{ml^4}}$$

2- Using equation (b)

$$P(\xi) = \omega^2 \int_\xi^l m\,\xi\, c\,\xi^2 d\xi = \frac{\omega^2 mc}{3}(l^3 - \xi^3)$$

Then the bending moment M(x)= $\int_x^l P(\xi)d\xi$

$$\therefore M(x) = \frac{\omega^2 mc}{3}\int_x^l (l^3 - \xi^3)d\xi = \frac{\omega^2 mc}{12}(3l^4 - 4l^3 + x^4)$$

$$\therefore U = \frac{1}{2EI}\int_0^l M_{(x)}^2 dx = \frac{1}{2EI}\int_0^l (\frac{\omega^2 mc}{12})^2 (3l^4 - 4l^3 + x^4)^2 dx$$

so

$$U_{max} = \frac{\omega^4}{2EI}\frac{m^2c^2}{144}\frac{312}{135}l^9$$

and the maximum kinetic energy

$$T_{max} = \frac{1}{2}\int_0^l \dot{y}^2\, m\, dx = \frac{1}{2}c^2 m\omega^2 \frac{l^5}{5}$$

From Rayleigh's quotient $U_{max} = T_{max}$

$$\omega = 3.53\sqrt{\frac{EI}{ml^4}}$$

6.2 Dunkerley's formula:

Dunkerley's formula gives the approximate value of the fundamental frequency of a composite system in terms of the natural frequencies of its component parts. It is derived by making use of the fact that the higher natural frequencies of most vibratory systems are large compared to their fundamental frequencies.

Let us take a 3DOF system . It's equation of motion is:

$$\{[K]-\omega^2[M]\}\{x\}=\{0\}$$

Or

$$\left\{[a][M]-\frac{1}{\omega^2}[I]\right\}=\{0\}$$

Or

$$\begin{vmatrix} a_{11}m_1-\frac{1}{\omega^2} & a_{12}m_2 & a_{13}m_3 \\ a_{21}m_1 & a_{22}m_2-\frac{1}{\omega^2} & a_{23}m_3 \\ a_{31}m_1 & a_{32}m_2 & a_{33}m_3-\frac{1}{\omega^2} \end{vmatrix}=0$$

Expanding this determinate , we obtain.

$$(\frac{1}{\omega^2})^3-(a_{11}m_1+a_{22}m_2+a_{33}m_3)(\frac{1}{\omega^2})^2+\cdots=0 \quad \dots\dots(1)$$

If the roots of this equation are $\frac{1}{\omega_1^2}$, $\frac{1}{\omega_2^2}$, $\frac{1}{\omega_3^2}$

Then equation (1) can be factored into following:

$$\left(\frac{1}{\omega^2}+\frac{1}{\omega_1^2}\right)\left(\frac{1}{\omega^2}+\frac{1}{\omega_2^2}\right)\left(\frac{1}{\omega^2}+\frac{1}{\omega_3^2}\right)=0$$

Or

$$(\frac{1}{\omega^2})^3-(\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}+\frac{1}{\omega_3^2})(\frac{1}{\omega^2})^2+\cdots=0$$

$From\ algebra$ we know that the sum of the roots of equation (1) must equal to the negative of the coefficient of its second term it also equal to the sum of the diagonal terms of matrix $[A]^{-1}$, which is called the trace of matrix.

$$trace[A]^{-1} = \sum_{i=1}^{3} \frac{1}{\omega_i^2}$$

For n-DOF

$$\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \cdots + \frac{1}{\omega_n^2} = a_{11}m_1 + a_{22}m_2 + \cdots + a_{nn}m_n \quad \ldots\ldots(2)$$

In most case , the higher frequencies $\omega_2, \omega_3, \ldots\ldots, \omega_n$ are considerably larger than the fundamental frequency ω_1 or

$$\frac{1}{\omega_1^2} \ll \frac{1}{\omega_i^2} \qquad i = 2, \ldots$$

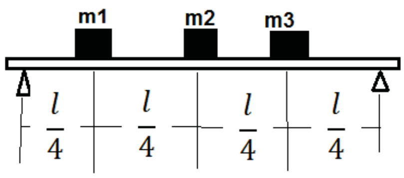
so we can say that for approximately

$$\frac{1}{\omega_1^2} \cong a_{11}m_1 + a_{22}m_2 + \cdots + a_{nn}m_n \qquad \ldots\ldots(3)$$

equation (3) known as Dunkerley's formula.

Example:

Estimate the fundamental natural frequency of a simple supported beam carrying three identical equally spaced masses as shown.$m_1 = m_2 = m_3 = m$



Solution :

From example 6.2 [Rao]

$$a_{11} = a_{33} = \frac{3}{256}\frac{l^3}{EI}$$

$$a_{22} = \frac{1}{48}\frac{l^3}{EI}$$

$$\frac{1}{\omega_1^2} \cong a_{11}m_1 + a_{22}m_2 + a_{33}m_3$$

$$\frac{1}{\omega_1^2} = \left(\frac{3}{256} + \frac{1}{48} + \frac{3}{256}\right)\frac{ml^3}{EI} \Rightarrow \omega_1 = 4.7537\sqrt{\frac{EI}{ml^3}}$$

$the\ Exact\ solution$

$\omega_1 = 4.93\sqrt{\frac{EI}{ml^3}}$ (using Rayleigh)

6.3The Rayleigh-Ritz method:

The Rayleigh-Ritz method can be consider an extension of Rayleigh method. It is based on the premise that a closer approximation to the exact natural modes can obtained by superposing (a number of assumed function)than by using a (single assumed function) as in Rayleigh method . if the assumed functions are suitably chosen, this method provides not only the approximate value of the fundamental frequency but also the approximate values of the higher natural frequencies and the mode shapes.

Starting with Rayleigh's equation.

$$\omega^2 = \frac{U_{max}}{T_{max}} \ldots\ldots\ldots\ldots\ldots (1)$$

Ritz assumed the deflection to be a sum of several function multiplied by constants. For a transverse vibration of beam.

$$y(x) = c_1\Phi_1(x) + c_2\Phi_2(x) + \ \ldots. + c_n\Phi_n(x)\ \ldots\ldots.. (2)$$

Where

$\Phi_i(x)$ are any admissible function satisfying the boundary conditions, since

$$\left.\begin{aligned} U &= \frac{1}{2}\sum_i\sum_j \mathrm{K_{ij}C_i\,C_j} \\ T &= \frac{1}{2}\ \sum_i\sum_j M_{ij}C_i\,C_j \end{aligned}\right\} \ldots\ldots. (3)$$

Where

For beam $K_{ij} = \int EI\,\ddot{\Phi}_i\ddot{\Phi}_j \mathrm{dx}$, $m_{ij} = \int m\Phi_i\,\Phi_j dx$

For longitudinal oscillation of slender rods

$$K_{ij} = \int EI\,\dot{\Phi}_i\dot{\Phi}_j \mathrm{dx}\ ,\ m_{ij} = \int m\Phi_i\,\Phi_j dx$$

Now minimize ω^2 by differentiating it with respect to each coefficient, and for C_i it can be seen as follows:

$$\frac{\partial \omega^2}{\partial c_i} = \frac{\partial}{\partial c_i}\left(\frac{U_{max}}{T_{max}}\right) = \frac{T_{max}\frac{\partial U_{max}}{\partial c_i} - U_{max}\frac{\partial T_{max}}{\partial c_i}}{T_{max}^2} = 0$$

Or

$$\frac{\partial U_{max}}{\partial c_i} - \frac{U_{max}}{T_{max}}\frac{\partial T_{max}}{\partial c_i} = 0$$

From equation (1)

$$\frac{\partial U_{max}}{\partial c_i} - \omega^2\frac{\partial T_{max}}{\partial c_i} = 0 \ldots\ldots..(4)$$

but

$\frac{\partial U_{max}}{\partial c_i} = \sum_j^n K_{ij}C_j$ and $\frac{\partial TU_{max}}{\partial c_i} = \sum_j^n M_{ij}C_j$

Substituting into equation (4), and rearranging in matrix form.

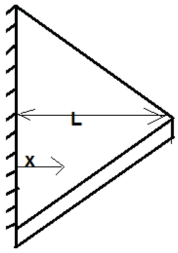
$$\sum_j^n K_{ij}C_j - \sum_j^n M_{ij}C_j = 0$$

$$\begin{bmatrix} (k_{11}-\omega^2 m_{11}) & (k_{12}-\omega^2 m_{12}) & \ldots & (k_{1n}-\omega^2 m_{1n}) \\ (k_{21}-\omega^2 m_{21}) & (k_{22}-\omega^2 m_{22}) & & \cdots \\ \vdots & & & \\ (k_{n1}-\omega^2 m_{n1}) & \cdots & & (k_{nn}-\omega^2 m_{nn}) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$$

The solution of this equation results in the n –natural frequencies. The mode shape is also be solving for C's for each natural frequency and substituting into equation (2) for the deflection.

Example:

A wedge shaped plate of constant thickness fixed into rigid wall as shown in figure. Determine the first two natural frequencies and mode shapes in longitudinal oscillation by using the Rayleigh-Ritz method using the two longitudinal modes.



$\Phi_1 = \sin\frac{\pi x}{2l}$, $\Phi_2 = \sin\frac{3\pi x}{2l}$

Or

u(x)= $c_1 \sin\frac{\pi x}{2l} + c_2 \sin\frac{3\pi x}{2l}$

$= c_1\Phi_1 + c_2\Phi_2$

m(x) = mass per unit length .

$= m_0\left(1-\frac{x}{l}\right)$

$EA(x) = stiffness = EA_0\left(1-\frac{x}{l}\right)$

$K_{ij} = \int_0^l EA(x)\,\dot{\Phi}_i\dot{\Phi}_j \mathrm{dx}$

$$m_{ij} = \int_0^l m(x)\Phi_i\,\Phi_j dx$$

$$K_{11} = \frac{\pi^2}{4l^2}\,EA_0\int_0^l\left(1-\frac{x}{l}\right)cos^2\frac{\pi x}{2l}\,dx = \frac{EA_0}{2l}\left(\frac{\pi^2}{8}+\frac{1}{2}\right) = 0.86685\,\frac{EA_0}{l}$$

$$K_{12} = \frac{3\pi^2}{4l^2}\,EA_0\int_0^l\left(1-\frac{x}{l}\right)cos\frac{\pi x}{2l}\,cos\frac{3\pi x}{2l}dx = 0.750\,\frac{EA_0}{l}$$

$$K_{22} = \frac{9\pi^2}{4l^2}\,EA_0\int_0^l\left(1-\frac{x}{l}\right)cos^2\frac{3\pi x}{2l}\,dx = \frac{EA_0}{2l}\left(\frac{9\pi^2}{8}+\frac{1}{2}\right) = 5.80165\,\frac{EA_0}{l}$$

$$m_{11} = m_0\int_0^l\left(1-\frac{x}{l}\right)sin^2\frac{\pi x}{2l}\,dx = m_0 l\left(\frac{1}{4}-\frac{1}{\pi^2}\right) = 0.148679 m_0 l$$

$$m_{12} = m_0\int_0^l\left(1-\frac{x}{l}\right)sin\frac{\pi x}{2l}\sin\frac{3\pi x}{2l}\,dx = m_0 l\left(\frac{1}{\pi^2}\right) = 0.101321 m_0 l$$

$$m_{22} = m_0\int_0^l\left(1-\frac{x}{l}\right)sin^2\frac{3\pi x}{2l}\,dx = m_0 l\left(\frac{1}{4}-\frac{1}{9\pi^2}\right) = 0.238742 m_0 l$$

Then

$$\begin{bmatrix} \left(0.86682\dfrac{EA_0}{l}-0.148679m_0l\omega^2\right) & \left(0.750\,\dfrac{EA_0}{l}-0.101321m_0l\omega^2\right) \\ \left(0.750\,\dfrac{EA_0}{l}-0.101321m_0l\omega^2\right) & \left(5.80165\dfrac{EA_0}{l}-0.238742m_0l\omega^2\right) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = 0$$

Now setting the determinant to Zero, the frequency equation can be obtained as:

$$\omega^4 - 36.3676\omega^2\alpha + 177.03\alpha^2 = 0$$

Where

$$\alpha = \frac{EA_0}{m_0 l^2}$$

So

$$\omega_1^2 = 5.789\alpha\ , \omega_2^2 = 30.5778\ \alpha$$

Using the results in matrix of C_1 & C_2

For mode 1 C_2=0.03689C_1 for mode 1 & $C_1 = 1$

For mode 2 C_1= -0.63819C_2 for mode 2 & $C_2 = 1$

The two natural frequencies are:

$\omega_1 = 2.4062\sqrt{\frac{EA_0}{m_0 l^2}}$ and

u_1(x)= $1.0\sin\frac{\pi x}{2l} + 0.03689\sin\frac{3\pi x}{2l}$

and

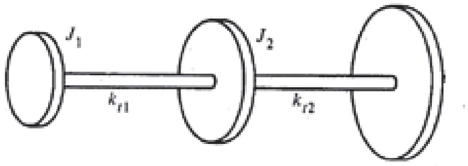
$\omega_2 = 5.5297\sqrt{\frac{EA_0}{m_0 l^2}}$ and

u_2(x)=$-0.63819\sin\frac{\pi x}{2l} + 1.0\sin\frac{3\pi x}{2l}$

6.4 **Holzer's Method for Torsional vibrations**:

Holzer 's method is basically a trial and error scheme to find the natural frequencies of un-damped semi definite vibrating system involving linear and angular displacement. In this method the trial frequency satisfies the constraints of the system.

Let us take the semi-definite system shown in the figure to illustrate the method.



The equation of the motion of the above torsional system are.

$J_1\ddot{\theta}_1 + k_{t1}(\theta_1 - \theta_2) = 0$ ……………..(1)

$J_2\ddot{\theta}_2 + k_{t1}(\theta_2 - \theta_1) + k_{t2}(\theta_2 - \theta_3) = 0$ ………(2)

$J_3\ddot{\theta}_3 + k_{t2}(\theta_3 - \theta_2) = 0$ ……………….(3)

Assume solution for harmonic motion

$\theta_i = \theta_{0i}\cos(\omega t - \emptyset)$ i=1,2,3

$\ddot{\theta}_i = -\omega^2\theta_i$

Substituting in equations (1),(2), and (3)

It can obtained that:

$\omega^2 J_1\theta_1 = k_{t1}(\theta_1 - \theta_2) \dots \dots \dots \dots \dots \dots \dots \dots \dots . \dots . (4)$

$\omega^2 J_2\theta_2 = k_{t1}(\theta_2 - \theta_1) + k_{t2}(\theta_2 - \theta_3) \dots \dots \dots . . (5)$

$\omega^2 J_3\theta_3 = k_{t2}(\theta_3 - \theta_2) \quad \dots \dots \dots \dots \dots \dots \dots \dots (6)$

Then the sum of the inertia torque (Mt) in the semi-definite system must be Zero.

Or

$\sum_{i=1}^{3}\omega^2 J_i\theta_i$=0 …………..(7)

In Holzer's method

1- a trial frequency ω is assumed , and θ_1 is arbitrarily chosen as unity.
2- $\theta_2$is computed from equation (4).

3- θ_3 is found from equation (5).

$Hence$

$\theta_1 = 1 \dots\dots\dots\dots\dots. (8)$

$\theta_2 = \theta_1 - \dfrac{\omega^2 J_1 \theta_1}{k_{t1}} \dots\dots\dots\dots\dots\dots (9)$

$\theta_3 = \theta_2 - \frac{\omega^2}{k_{t2}} \, (J_1\theta_1 + J_2\theta_2) \dots\dots(10)$

The values are substituted in equation (7) to verify whether the constrains is satisfied . if equation

(7) is not satisfied , a new trial value of $\omega$is assumed and the procedure will repeated.

Equations 7,8,9 and 10 can be generalized for n-disc system as follows:

$\sum_{\mathrm{i}=1}^{\mathrm{n}} \omega^2 J_i \theta_i$=0…………..(11)

$\theta_i = \theta_{i-1} - \frac{\omega^2}{k_{i-1}} \, (\sum_{k=1}^{i-1} J_k \, \theta_k)$…….(12) i=2,3,…,n

Although Holzer's method can be applied for the vibration of spring-mass system. In the same procedure it can be obtained that:

$X_i = X_{i-1} - \frac{\omega^2}{k_{i-1}} \, (\sum_{k=1}^{i-1} m_k \, x_k)$…….(13) i=2,3,…,n

and

F=$\sum_{\mathrm{i}=1}^{\mathrm{n}} \omega^2 m_i x_i$

The trial will continue until F will be equal zero in this system and

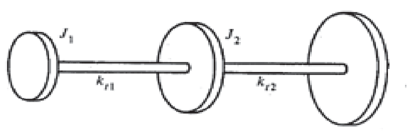
$M_t = 0$ for torsional systems.

Example:

Determine the natural frequencies and mod shapes of the system shown in figure.

For $J_1 = 5 \; kg.m^2, J_2 = 11 kg.m^2 \; , and \; J_3 = \; 22 \; kg.m^2$

$k_{t1} = 0.1 \, x10^6 \; N.\dfrac{m}{rad}$, $\quad k_{t2} = 0.2 \, x10^6 \; N.\dfrac{m}{rad}$



Solution:

For making the procedures more easer the following table could be used:

Parameters of the system

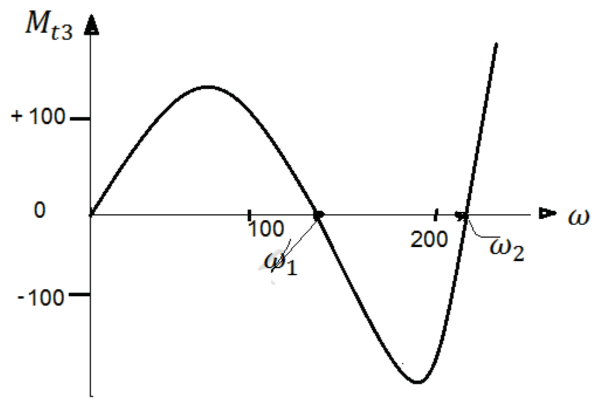
Station 1	Station 2	Station 3
$J_1 = 5$ $k_{t1} = 0.1\,x10^6$	$J_2 = 11$ $k_{t2} = 0.2\,x10^6$	$J_3 = 22$ $k_{t3} = 0$

Calculation Program

ω ω^2	$\theta_1 = 1.0$ $M_{t1}=\omega^2\theta_1 J_1$	$\theta_2 = 1 - {M_{t1}}/{k_{t1}}$ $M_{t2}=M_{t1}+\omega^2\theta_2 J_2$	$\theta_3 = \theta_2 - {M_{t2}}/{k_{t2}}$ $M_{t3} = M_{t2}+\omega^2\theta_3 J_3$
20 400	1.0 2.0 x 10^3	0.980 6.312 x 10^3	0.9484 14.66 x 10^3
40 1600	1.0 8.0 x 10^3	0.920 24.19 x 10^3	0.799 52.32 x 10^3

M_{t3} is the moment (torque) to the right of disk 3, which must be Zero at the natural frequencies.

And to get the values of the system natural frequencies it is recommended To plot a curve between $M_{t3}\,x10^{-3}$ & ω as follows:



From the plot $\omega_1 = 123.666$, $\omega_2 = 202.658$

THE END OF THE LECTURE NOTES IN ADVANCE VIBRATIONS