These equations show that the strain approximation within the element allows $\epsilon_x$ to be a linear function of *y* and $\epsilon_y$ to be a linear function of *x*, while $\gamma_{xy}$ is a linear function in *x* and *y*. So by the addition of one node we have gained a much better approximate solution within the element field for the general case where the strains vary with both *x* and *y* throughout the structure domain. Since the stress-strain relations are constant, the stress components may vary similarly within the element field.

Also, in this case the function satisfies compatibility within the element because the function is continuous. Along the element edges for *x* = constant or *y* = constant the displacement takes a linear form, and thus remains a straight line between any two of the corner nodes. Therefore element connections to other elements satisfy compatibility as long as corner nodes of one element connect to the corner nodes of the adjacent element. Connection of two adjacent elements to a third element such that the edge of the third element spans two of the adjacent elements' edges, as shown in Figure 4-3, violates compatibility.

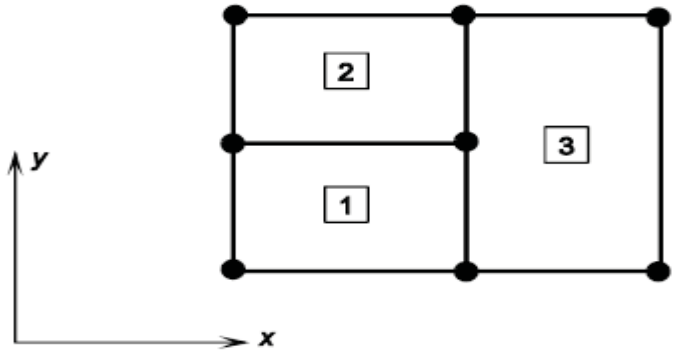



*Figure 4-3. Two-Dimensional Element Compatibility Violation*

Before continuing the discussion of the square element, make a change of coordinate systems to transform the square into a quadrilateral shape. Square elements can geometrically model very few structures. A co-ordinate transformation from *x,y* to $\xi,\eta$ produces the quadrilateral element sketched in Figure 4-4. We call the element an *isoparametric quadrilateral* because the same interpolation functions (parameters) used to define the displacement field define the geometric transformation.

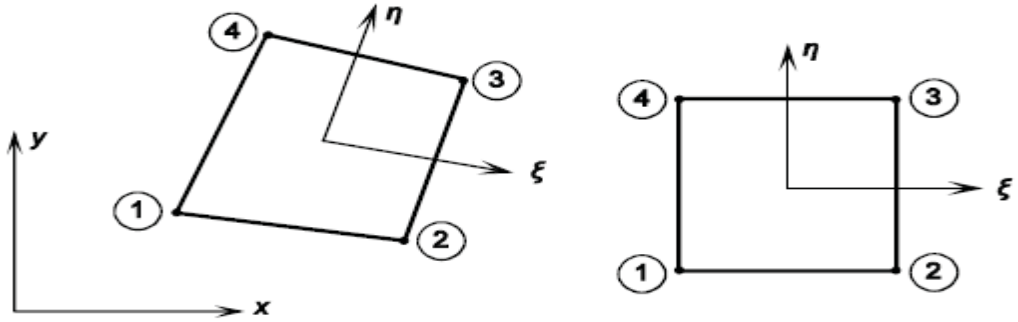



*Figure 4-4. Two-Dimensional Quadrilateral Element*

# Lecture 4: EigenValue Problems

EigenValue, or characteristics- Value, Problems are Special class of Boundary- Value problems that are common in engineering problems contexts involving ( Vibrations, Elasticity, & other Oscillating systems).

## 1. Mathematical Background

Consider
$$q_{11}x_1 + q_{12}x_2 + \dots + q_{1n}x_n = 0$$
$$q_{21}x_1 + q_{22}x_2 + \dots + q_{2n}x_n = 0$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$q_{n1}x_1 + q_{n2}x_2 + \dots + q_{nn}x_n = 0 \qquad (1)$$

$q_{ij}$ *may involve ( constants, variables, &/or functions).* Obviously the set has *a trivial solution*, namely $x_1 = x_2 = \dots = x_n = 0$. This solution is of no practical to engineer.

*A nontrivial solution* is required if physical problems are to be mathematically modeled, this is precisely the essence of the ***Eigen Problems***; for the set (1) we have

$$x_j = \frac{|Qj|}{|Q|}\ ;\ j = 1, 2, \dots, n \qquad (2)$$

Where: $|Qj|$ is the determinant of the $q_{ij}$ coefficient matrix whose $j^{th}$ column is replaced by the null vector; $|Q|$ is the determinant of the $q_{ij}$ coefficient matrix, & $x_j$ is the value of the $j^{th}$ variable. $|Qj|$ is always zero, because the *null vector* will always appear as a column in the matrix $|Qj|$.

Thus $|Q|\,x_j = 0$, j= 1, 2, ….., n (3)

if $x_j \neq 0$ then $|Q| \rightarrow$ zero. This condition must be satisfied in ***Eigen Problems***, if a nontrivial solution is to be attained. That is $|Q| = 0$ (4)

## 2. Characteristic equation Determination

For expanding the determinant of $[Q]$matrix as is required by eq. (4), generally, $[Q]$ matrix is assumed to be related to other two matrices as: $[Q] = [A] - [B]$ (5)

Where $[A]$ is composed of *Constant elements* & $[B]$is composed of *Constants, Variables, or Both elements*.

In practice, we may encounter one of three classes of ***Eigen problems***.

a. **Type I**; where $[B]$ in eq. (5) is composed of elements some or all of which are functions of a variables as :

$$[Q1]\{X\} = [[A] - [B]]\{X\} = \{0\} \qquad (6)$$

where
$$[B] = \begin{bmatrix} f11(\lambda) & \dots & f1n(\lambda) \\ . & .. & . \\ fn1(\lambda) & \dots & fnn(\lambda) \end{bmatrix}$$

Clearly, $f_{11}(\lambda), \dots, f_{nn}(\lambda)$ may take on any value including zero.

b. **Type II;** where $[B]$ in eq. (5) is equal to the product of the variable λ and a composed matrix of constant elements $[C]$; thus

$$[Q2]\{X\} = [[A] - \lambda[C]]\{X\} = \{0\} \qquad (7)$$

c. **Type III;** this is the ***simplest Eigen problem***. Here the $[B]$ matrix is equal to the product of the variable λ and the Identity matrix $[I]$; as

$$[Q3]\{X\} = [[A] - \lambda[I]]\{X\} = \{D\} \qquad (8)$$

Note that in all three types, the $[A]$ matrix contains scalar coefficients. Furthermore, the Eigen problems given by eq. (7) can be easily reduced to that given by eq. (8).{ ***Home Work you have to do it***}. For some applications the form that developed by you $\{[[D] - \lambda[I]]\{X\} = \{D\}\}$ might be more appropriate to use than that given by eq. (7).

**3.** In other wards the **Mathematical Background can be** discussed by:

for solving sets of linear algebraic equations of the general form

$$[A]\{X\}=\{B\}$$

Such systems are called *non homogeneous* because of the presence of the vector $\{B\}$ on the right-hand side of the equality. If the equations comprising such a system are linearly independent (that is, have a nonzero determinant), they will have a unique solution. In other words, there is one set of x values that will make the equations balance.

In contrast, a homogeneous linear algebraic system has the general form

$$[A]\{X\}=0$$

Although nontrivial solutions (that is, solutions other than all x's = 0) of such systems are possible, they are generally not unique. Rather, the simultaneous equations establish relationships among the x's that can be satisfied by various combinations of values.

Eigenvalue problems associated with engineering are typically of the general form

$$\begin{aligned}(a_{11}-\lambda)x_1 + \quad & a_{12}x_2 + \cdots + \quad a_{1n}x_n = 0\\ a_{21}x_1 + \quad & (a_{22}-\lambda)x_2 + \cdots + \quad a_{2n}x_n = 0\\ & \cdots\\ a_{n1}x_1 + \quad & a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n = 0\end{aligned}$$

where $\lambda$ is an unknown parameter called the *eigenvalue,* or *characteristic value.* A solution $\{X\}$ for such a system is referred to as an *eigenvector.* The above set of equations may also be expressed concisely as

$$[[A]-\lambda[I]]\{X\}=0 \qquad (8')$$

The solution of Eq. (8') hinges on determining $\lambda$. One way to accomplish this is based on the fact that the determinant of the matrix [[A] - $\lambda$ [ I ] ] must equal zero for nontrivial solutions to be possible. Expanding the determinant yields a polynomial in $\lambda$ . The roots of this polynomial are the solutions for the eigenvalues. An example of this approach will be provided in the next section.

**Physical Background**

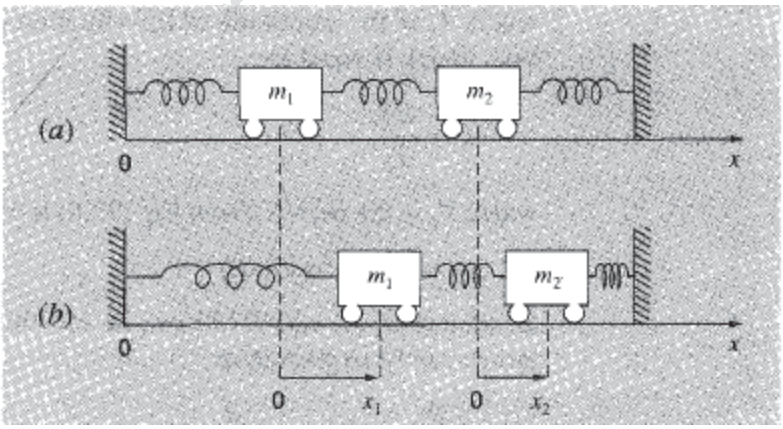



**Figure (1)**

Positioning the masses away from equilibrium creates forces in the springs that upon release lead to oscillations of the masses. The positions of the masses can be referenced to local coordinates with origins at their respective equilibrium positions.

The mass-spring system in Fig. *(1a)* is a simple context to illustrate how eigenvalues occur in physical problem settings. It also will help to illustrate some of the mathematical concepts introduced in the previous section.

To simplify the analysis, assume that each mass has no external or damping forces acting on it. In addition, assume that each spring has the same natural length 1 and the same spring constant k. Finally, assume that the displacement of each spring is measured relative to its own local coordinate

system with an origin at the spring's equilibrium position (Fig. 1 b). Under these assumptions, Newton's second law can be employed to develop a force balance for each mass ,

$$m_1\frac{d^2x_1}{dt^2} = -kx_1 + k(x_2 - x_1)$$

$$m_2\frac{d^2x_2}{dt^2} = -k(x_2 - x_1) - kx_2$$

and

where $x_i$ is the displacement of mass i away from its equilibrium position (Fig. 1 b).

These equations can be expressed as

$$m_1\frac{d^2x_1}{dt^2} - k(-2x_1 + x_2) = 0 \qquad (9\ a)$$

$$m_2\frac{d^2x_2}{dt^2} - k(x_1 - 2x_2) = 0 \qquad (9\ b)$$

From vibration theory, it is known that solutions to Eq. (9) can take the form

$$x_i = A_i \sin(\omega t) \qquad (10)$$

where $A_i$ = the amplitude of the vibration of mass ***i*** and ω = the frequency of the vibration, which is equal to

$$\omega = \frac{2\pi}{Tp} \qquad (11)$$

where $T_p$ is the period. From Eq. (10) it follows that

$$x_i'' = -A_i\omega^2 \sin(\omega t) \qquad (12)$$

Equations (10) and (12) can be substituted into Eq. (9), which, after collection of terms, can be expressed as

$$\left(\frac{2k}{m_1} - \omega^2\right)A_1 - \frac{k}{m_1}A_2 = 0 \qquad (13\ a)$$

$$-\frac{k}{m_2}A_1 + \left(\frac{2k}{m_2} - \omega^2\right)A_2 = 0 \qquad (13\ b)$$

Comparison of Eq. (13) with Eq. (8' ) indicates that at this point, the solution has been reduced to an eigenvalue problem.

**Example 1: Eigenvalues and Eigenvectors for a Mass-Spring System**

**Problem Statement**. Evaluate the eigenvalues and the eigenvectors of Eq. (13) for the case where $m_1 = m_2 = 40$ kg and k = 200 N/m.

Solution. Substituting the parameter values into Eq. (13) yields

$$(10 - \omega^2)A_1 - 5A_2 = 0$$

$$-5A_1 + (10 - \omega^2)A_2 = 0$$

The determinant of this system is

$$(\omega^2)^2 - 20\omega^2 + 75 = 0$$

***{ Note that the value of the second-order determinant*** $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ ***is calculated by*** $D = a_{11}a_{22} - a_{12}a_{21}$ ***}***

which can be solved by the quadratic formula for $w^2 = 15$ and $5\ s^{-2}$ .Therefore, the frequencies for the vibrations of the masses are $w = 3.873\ s^{-1}$ and $2.236\ s^{-1}$, respectively. These values can be used to determine the periods for the vibrations with Eq. (11). For the first mode, $T_p = 1.62\ s$ and for the second, $T_p = 2.81\ s$.

As stated above, a unique set of values cannot be obtained for the unknowns. However, their ratios can be specified by substituting the eigenvalues back into the equations. For example, for the first mode ($w^2 = 15\ s^{-2}$) , $A_1 = -A_2$. For the second mode ($w^2 = 5\ s^{-2}$), $A_1 = A_2$.

This example provides valuable information regarding the behavior of the system in Fig. 1. Aside from its period, we know that if the system is vibrating in the first mode, the amplitude of the second mass will be equal but of opposite sign to the amplitude of the first. As in Fig. 2 a, the masses vibrate apart and then together indefinitely.

In the second mode, the two masses have equal amplitudes at all times. Thus, as in Fig. 2 b, they vibrate back and forth in unison. It should be noted that the configuration of the amplitudes provides guidance on how to set their initial values to attain pure motion in either of the two modes. Any other configuration will lead to superposition of the modes.

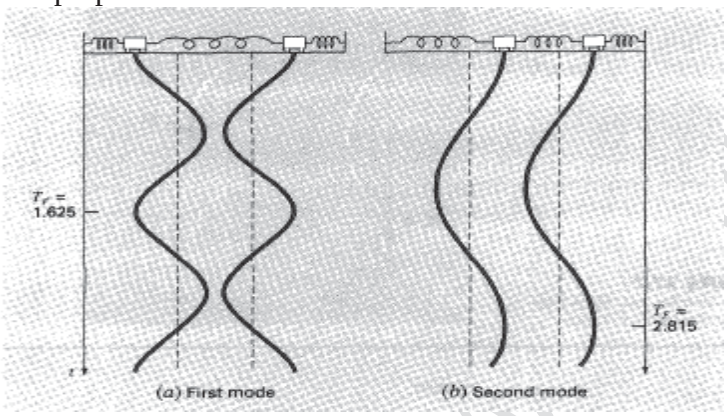



**Figure (2)**

The principal modes of vibration of two equal masses connected by three identical springs between fixed walls.

4. **A boundary Value Problem for OD eq.s.**

Now that you have been introduced to eigenvalues, we turn to the type of problem that is the subject of the present chapter: boundary-value problems for ordinary differential equations. **Figure 3**shows a physical system that can serve as a context for examining this type of problem.

The curvature of a slender column subject to an axial load ***P*** can be modeled by

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \qquad (14)$$

where $d^2y/dx^2$ specifies the curvature, $M$ = the bending moment, $E$ = the modulus of elasticity, and $I$ = the moment of inertia of the cross section about its neutral axis. Considering the free body in Fig. *3b*, it is clear that the bending moment at $x$ is $M = -Py$. Substituting this value into Eq. (14) gives

$$\frac{d^2y}{dx^2} + p^2y = 0 \qquad (15)$$

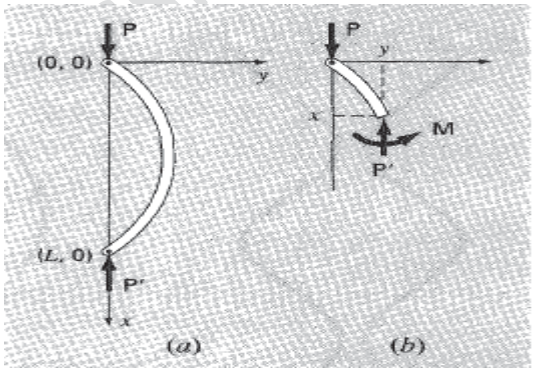



**Figure (3)**

(a) A slender rod. *(b)* A free body diagram of a rod.

where $$p^2 = \frac{P}{El} \qquad (16)$$

For the system in Fig.3 , subject to the boundary conditions

$$y(0) = 0 \qquad (17\ a) \ ; \qquad y(L) = 0 \qquad (17\ b)$$

the general solution for Eq. *(15)* is

$$y = A \sin (px) + B \cos (px) \qquad (18)$$

where *A* and B are arbitrary constants that are to be evaluated via the boundary conditions. According to the first condition [Eq. *(17a)*],

$$0 = A \sin (0) + B \cos (0)$$

Therefore, we conclude that $B = 0$.

According to the second condition [Eq. *(17 b)*],

$$0 = A \sin (pL) + B \cos (pL)$$

But, since $B = 0$, $\rightarrow A \sin (pL) = 0$. Because $A = 0$ represents a *trivial solution*, we conclude that sin $(pL) = 0$. For this equality to hold,

$$pL = n\pi \qquad \text{for n} = 1 , 2 , 3 , \ldots \qquad (19)$$

Thus, there are an infinite number of values that meet the boundary condition. Equation *(19)* can be solved for

$$p = \frac{n\pi}{L} \qquad \text{for n} = 1 , 2 , 3 , \ldots \qquad (20)$$

which are the eigenvalues for the column.

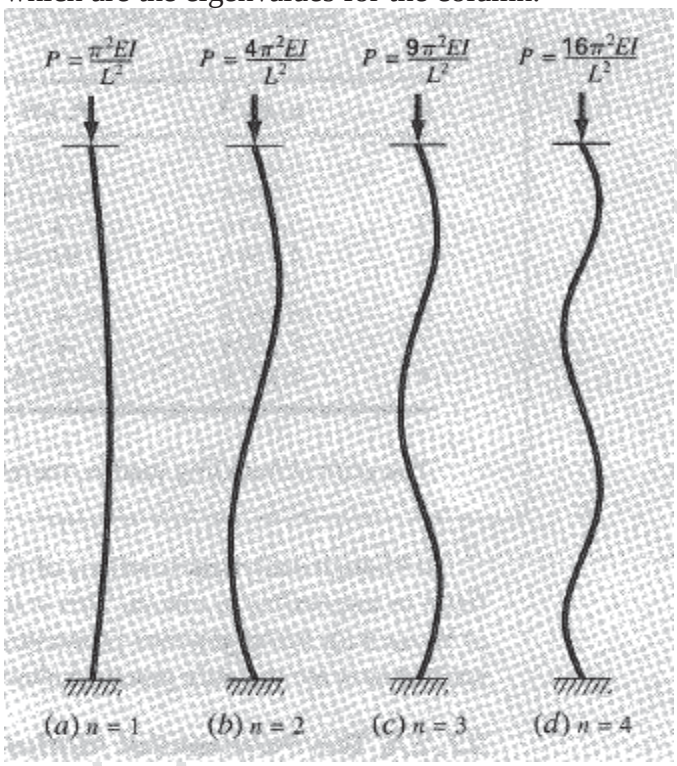



**Figure (4)**

The first four eigenvalues for the slender rod from Fig. 3.

Figure 4, which shows the solution for the first four eigenvalues, can provide insight into the physical significance of the results. Each eigenvalue corresponds to a way in which the column buckles. Combining Eqs. (16) and (20) gives

$$P = \frac{n^2\pi^2 EI}{L^2} \qquad \text{for n} = 1 , 2 , 3 , \ldots \qquad (21)$$

These can be thought of as *buckling loads* because they represent the levels at which the column moves into each succeeding buckling configuration. In a practical sense, it is usually the first value that is of interest because failure will usually occur when the column first buckles. Thus, a critical load can be defined as

$$P = \frac{\pi^2 EI}{L^2}$$ which is formally known as *Euler's formula.*

**Example 2: Eigenvalue Analysis of an Axially Loaded Column**

**Problem Statement**. An axially loaded wooden column has the following characteristics: $E$ =10xl0$^9$ Pa, $I$ = 1.25 x 10$^{-5}$ m$^4$, and $L$ = 3 m. Determine the first eight eigenvalues and the corresponding buckling loads.

Solution. Equations (20) and (21) can be used to compute

| n | p, m$^{-2}$ | P, kN |
|---|---|---|
| 1 | 1.0472 | 137.078 |
| 2 | 2.0944 | 548.311 |
| 3 | 3.1416 | 1233.701 |
| 4 | 4.1888 | 2193.245 |
| 5 | 5.2360 | 3426.946 |
| 6 | 6.2832 | 4934.802 |
| 7 | 7.3304 | 6716.814 |
| 8 | 8.3776 | 8772.982 |

The critical buckling load is, therefore, 137.078 kN.

Although analytical solutions of the sort obtained above are useful, they are often difficult or impossible to obtain. This is usually true when dealing with complicated systems or those with heterogeneous properties. In such cases, numerical methods of the sort described next are the only practical alternative.

5. **The Polynomial Method**

Equation (15) can be solved numerically by substituting a central finite-divided difference approximation for the second derivative to give

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + p^2 y_i = 0$$

which can be expressed as $$y_{i-1} - (2 - h^2p^2)y_i + y_{i+1} = 0 \qquad (22)$$

Writing this equation for a series of nodes along the axis of the column yields a homogeneous system of equations. For example, if the column is divided into five segments (that is, four interior nodes), the result (***of eq. 22 multiplied by -1***) is

$$\begin{bmatrix} (2-h^2p^2) & -1 & 0 & 0 \\ -1 & (2-h^2p^2) & -1 & 0 \\ 0 & -1 & (2-h^2p^2) & -1 \\ 0 & 0 & -1 & (2-h^2p^2) \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = 0 \qquad (23)$$

Expansion of the determinant of the system yields a polynomial, the roots of which are the eigenvalues. This approach, called the *polynomial method,* is performed in the following example.

**EXAMPLE 3: The Polynomial Method**

**Problem Statement**. Employ the polynomial method to determine the eigenvalues for the axially loaded column from **Example 2** using *(a)* one, (b) two, (c) three, and *(d)* four interior nodes.

**Solution.**

(a) Writing Eq. *(22)* for one interior node yields *(h = 3/2)*

$$-(2 - 1.5^2p^2)y_i = 0$$
$$-(2 - 2.25p^2)y_i = 0$$

Thus, for this simple case, the eigenvalue is analyzed by setting the determinant equal to zero

$$(2 - 2.25p^2) = 0 \Longrightarrow p = \pm 0.9428$$

which is about *10* percent less than the exact value of *1.0472* obtained in Example *2.*

(b) For two interior nodes *(h = 3/3),* Eq. *(22)* is written as

$$\begin{bmatrix}(2-h^2p^2) & -1\\ -1 & (2-h^2p^2)\end{bmatrix}\begin{Bmatrix}y_1\\ y_2\end{Bmatrix}=0$$

Expansion of the determinant gives

$$(2-p^2)^2-1=0$$

which can be solved for $p = \pm 1$ and $\pm 1.73205$. Thus, the first eigenvalue is now about *4.5* percent low and a second eigenvalue is obtained that is about *17* percent low.

(c) For three interior points *(h = 3/4),* Eq. *(22)* yields

$$\begin{bmatrix}2-0.5625p^2 & -1 & 0\\ -1 & 2-0.5625p^2 & -1\\ 0 & -1 & 2-0.5625p^2\end{bmatrix}\begin{Bmatrix}y_1\\ y_2\\ y_3\end{Bmatrix}=0$$

The determinant can be set equal to zero and expanded to give

$$(2-0.5625p^2)^3-2(2-0.5625p^2)=0$$

For this equation to hold, $2 - 0.5625p^2 = 0$ and $2 - 0.5625p^2 = \sqrt{2}$. Therefore, the first three eigenvalues can be determined as

$p = \pm 1.0205$ $\quad |\varepsilon_t| = 2.5\%$

$p = \pm 1.8856$ $\quad |\varepsilon_t| = 10\%$

$p = \pm 2.4637$ $\quad |\varepsilon_t| = 22\%$ where exact $p_3=3.1416$

**(d)** For four interior points *( h = 3/5) ,* the result is Eq. *(23)* with $2 - 0.36p^2$ on the diagonal. Setting the determinant equal to zero and expanding it gives

$$(2-0.36p^2)^4-3(2-0.36p^2)^2+1=0$$

which can be solved for the first four eigenvalues

$p = \pm 1.0301$ $\quad |\varepsilon_t| = 1.6\%$

$p = \pm 1.9593$ $\quad |\varepsilon_t| = 6.5\%$

$p = \pm 2.6967$ $\quad |\varepsilon_t| = 14\%$

$p = \pm 3.1702$ $\quad |\varepsilon_t| = 24\%$

**Table 2**

**The results of applying the polynomial method to an axially loaded column. The numbers in parentheses represent the absolute value of the true percent relative error.**

| | | Polynomial Method | | | |
|---|---|---|---|---|---|
| **Eigenvalue** | **True** | **h = 3/2** | **h = 3/3** | **h = 3/4** | **h = 3/5** |
| 1 | 1.0472 | 0.9428 (10%) | 1.0000 (4.5%) | 1.0205 (2.5%) | 1.0301 (1.6%) |
| 2 | 2.0944 | | 1.7321 (21%) | 1.8856 (10%) | 1.9593 (6.5%) |
| 3 | 3.1416 | | | 2.4637 (22%) | 2.6967 (14%) |
| 4 | 4.1888 | | | | 3.1702 (24%) |

Table 2, which summarizes the results of this example, illustrates some fundamental aspects of the polynomial method. As the segmentation is made more refined, additional eigenvalues are determined and the previously determined values become progressively more accurate. Thus, the approach is best suited for cases where the lower eigenvalues are required.

**6.** The **Power Method ( Vector Iteration Techniques )**

The power *method* is an iterative approach that can be employed to determine the largest eigenvalue. With slight modification, it can also be employed to determine the smallest and the intermediate

values. ***It can be applied to eigenproblems of Types II & III, but not to type I***. It has the additional benefit that the corresponding eigenvector is obtained as a by-product of the method.

a) **Determination of the Largest Eigenvalue.** To implement the power method, the system being analyzed must be expressed in the form

$$[[A] - \lambda[B]]\{X\} = \{0\} \rightarrow [A]\{x\} = \lambda[B]\{x\} \rightarrow [B]^{-1}[A]\{x\} = \lambda\{x\} \rightarrow or\ in\ the\ equivelant$$
$$[C]\{x\} = \lambda\{x\} \quad (24)$$

In general, eq. (24) can be expressed in an iterative form as

$$[C]\{x\}_i = \lambda\{x\}_{i+1} \quad (25)$$

If a trial vector $\{x\}_0 \neq \{0\}$ , ***( normally $\{x\}_0 = \{1\}$ )*** , was assumed then a new vector $\{x\}_1$ can be calculated, as follows

$$[C]\{x\}_0 = \lambda_1\{x\}_1$$
$$[C]\{x\}_1 = \lambda_2\{x\}_2$$

.....

$$[C]\{x\}_{k-1} = \lambda\{x\}_k$$

**EXAMPLE 4: Power Method for Highest Eigenvalue**

**Problem Statement**. Employ the power method to determine the highest eigenvalue for part *(c)* of **Example 3.**

**Solution**. $h^2 = 0.75^2 = 0.5625$

in 1st row of example 3-c → $\frac{2}{0.5625}$ = 3.556; $\frac{-1}{0.5625}$ = -1.778 & so on.

The system is first written in the form of Eq. (25) as

$$3.556\,x_1 - 1.778x_2 = \lambda x_1$$
$$-1.778x_1 + 3.556\,x_2 - 1.778x_3 = \lambda x_2$$
$$-1.778x_2 + 3.556\,x_3 = \lambda x_3$$

Then, assuming the x's on the left-hand side of the equation arc equal to I $\{x_i\}^T = \{1,1,1\}^T$,

$$3.556(1) - 1.778(1) = 1.778 = \lambda x_1$$
$$-1.778(1) + 3.556(1) - 1.778(1) = 0 = \lambda x_2$$
$$-1.778(1) + 3.556(1) = 1.778 = \lambda x_3$$

in matrix form $[C]\{x\} = \lambda\{x\}$ gives;

1 st itteration

$$\begin{bmatrix} 3.556 & -1.778 & 0 \\ -1.778 & 3.556 & -1.778 \\ 0 & -1.778 & 3.556 \end{bmatrix}\begin{Bmatrix}\mathbf{1}\\\mathbf{1}\\\mathbf{1}\end{Bmatrix} = \begin{Bmatrix}\mathbf{1.778}\\\mathbf{0}\\\mathbf{1.778}\end{Bmatrix} = \lambda_1\{x\}_1 = 1.778\begin{Bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{1}\end{Bmatrix}$$

2 nd iteration

$$\begin{bmatrix} 3.556 & -1.778 & 0 \\ -1.778 & 3.556 & -1.778 \\ 0 & -1.778 & 3.556 \end{bmatrix}\begin{Bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{1}\end{Bmatrix} = \begin{Bmatrix}3.556\\-3.556\\3.556\end{Bmatrix} = \lambda_2\{x\}_2 = 3.556\begin{Bmatrix}\mathbf{1}\\\mathbf{-1}\\\mathbf{1}\end{Bmatrix}$$

after 4th & 5th iterations those gives;

$\lambda_4$= 6.223 & $\lambda_5$= 6.095= λ with error percent of $|\varepsilon_a| = \left|\frac{6.095-6.223}{6.223}\right|$ x 100%= 2.06%

→→ The highest EigenValue is 6.095 ≈ $p^2 = (p_3)^2 = 2.4637^2 = 6.070$

**b) Power Method for Lowest Eigenvalue**

This problem may arise when engineer is dealing with frictional materials such as soils. The basic criterion used in determining the state of stress for such materials is based on the so called Mohr failure criteria in which both the maximum principal stress $\sigma_1$ & minor principal stress $\sigma_3$ are needed. Considering the basic eigenvalue equation;

$[[A] - \lambda[B]]\{X\} = \{0\}$ or $[A]\{x\} = \lambda[B]\{x\}$ (26)

Pre multiplying eq. 26 by $[A]^{-1}$ gives

$[A]^{-1}[A]\{x\} = [A]^{-1}\lambda[B]\{x\}$ →→ $[I]\{x\} = [A]^{-1}\lambda[B]\{x\}$ dividing by λ gives

$$[[A]^{-1}[B]]\{x\} = \frac{1}{\lambda}\{x\}_{i+1}\ ;\ i= 1, 2,..., k \qquad (27)$$

For special case when $[B] = [I]$ eq. 27 becomes;

$$[A]^{-1}\{x\} = \frac{1}{\lambda}\{x\}_{i+1} \qquad (27')$$

i.e. when λ has the highest then $\frac{1}{\lambda}$ becomes the smallest

**EXAMPLE 5: Power Method for Lowest Eigenvalue**

**Problem Statement**. Employ the power method to determine the lowest eigenvalue for part (c) of Example 3.

Solution. Recall from Example 4

$$[A] = \begin{bmatrix} 3.556 & -1.778 & 0 \\ -1.778 & 3.556 & -1.778 \\ 0 & -1.778 & 3.556 \end{bmatrix}$$ the matrix inverse is $$[A]^{-1} = \begin{bmatrix} 0.422 & 0.281 & 0.141 \\ 0.281 & 0.562 & 0.281 \\ 0.141 & 0.281 & 0.422 \end{bmatrix}$$

Using the same format as in Example 4, the power method can be applied to this matrix.

*First iteration:*

$$\begin{bmatrix} 0.422 & 0.281 & 0.141 \\ 0.281 & 0.562 & 0.281 \\ 0.141 & 0.281 & 0.422 \end{bmatrix}\begin{Bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0.884} \\ \mathbf{-1.124} \\ \mathbf{0.884} \end{Bmatrix} = 1.124\begin{Bmatrix} \mathbf{0.751} \\ \mathbf{1} \\ \mathbf{0.751} \end{Bmatrix}$$

*Second iteration:*

$$\begin{bmatrix} 0.422 & 0.281 & 0.141 \\ 0.281 & 0.562 & 0.281 \\ 0.141 & 0.281 & 0.422 \end{bmatrix}\begin{Bmatrix} \mathbf{0.751} \\ \mathbf{1} \\ \mathbf{0.751} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0.704} \\ \mathbf{0.984} \\ \mathbf{0.704} \end{Bmatrix} = \mathbf{0.984}\begin{Bmatrix} \mathbf{0.715} \\ \mathbf{1} \\ \mathbf{0.715} \end{Bmatrix}$$

where $|\varepsilon_a|$= 14.6%.

*Third iteration:*

$$\begin{bmatrix} 0.422 & 0.281 & 0.141 \\ 0.281 & 0.562 & 0.281 \\ 0.141 & 0.281 & 0.422 \end{bmatrix}\begin{Bmatrix} \mathbf{0.715} \\ \mathbf{1} \\ \mathbf{0.715} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0.684} \\ \mathbf{0.964} \\ \mathbf{0.684} \end{Bmatrix} = \mathbf{0.964}\begin{Bmatrix} \mathbf{0.709} \\ \mathbf{1} \\ \mathbf{0.709} \end{Bmatrix}$$

where $|\varepsilon_a|$= 4%.

Thus, after only three iterations, the result is converging on the value of 0.955, which , is the reciprocal of the smallest eigenvalue, 1.0472, obtained in Example 3.

c) **Determination of Intermediate Eigenvalues**. After finding the largest eigenvalue, it is possible to determine the next highest by replacing the original matrix by one that includes only the remaining eigenvalues. The process of removing the largest known eigenvalue is called *deflation.* The technique outlined here, *Hotelling's method,* is designed for symmetric matrices. This is because it exploits the orthogonality of the eigenvectors of such matrices, which can be expressed as

$$\{x\}_i^T\{x\}_j = \begin{cases} 0 & for\ i \neq j \\ 1 & for\ i = j \end{cases} \qquad (28)$$

where the components of the eigenvector {X} have been normalized so that $\{x\}^T\{x\}$ =1; that is, so that the sum of the squares of the components equals 1. This can be accomplished by dividing each of the elements by the normalizing factor

$$\sqrt{\sum_{k=1}^{n} x_k^2}$$

Now, a new matrix $[A]_2$ can be computed as

$$[A]_2 = [A]_1 - \lambda_1 \{x\}_1^T \{x\}_1 \qquad (29)$$

where $[A]_1$ = the original matrix and $\lambda_1$ = the largest eigenvalue. If the power method is applied to this matrix, the iteration process will converge to the second largest eigenvalue, $\lambda_2$ . To show this, first post multiply Eq. (29) by $\{X\}_1$,

$$[A]_2 \{x\}_1 = [A]_1 \{x\}_1 - \lambda_1 \{x\}_1 \{x\}_1^T \{x\}_1$$

Invoking the orthogonality principle converts this equation to

$$[A]_2 \{x\}_1 = [A]_1 \{x\}_1 - \lambda_1 \{x\}_1$$

where the right-hand side is equal to zero according to Eq. (24). Thus, $[A]_2 \{x\}_1 = 0$. Consequently, $\lambda = 0$ and $\{X\} = \{x\}_1$ is a solution to $[A]_2 \{X\} = \lambda\{X\}$. In other words, the $[A]_2$ has eigenvalues of *0*, $\lambda_2, \lambda_3, \ldots, \lambda_n$ . The largest eigenvalue, $\lambda_1$, has been replaced by a 0 and, therefore, the power method will converge on the next biggest $\lambda_2$.

The above process can be repeated by generating a new matrix $[A]_3$, etc. Although in theory this process could be continued to determine the remaining eigenvalues, it is limited by the fact that errors in the eigenvectors are passed along at each step. Thus, it is only of value in determining several of the highest eigenvalues. Although this is somewhat of a shortcoming, such information is precisely what is required in many engineering problems.

**Algorithm of Determination of Intermediate Eigenvalues**

a) Compute the largest eigenvalue and its corresponding vector ( $\lambda_1$ , $\{x\}_1$ );

b) Normalize the eigenvector $\{x\}_1$ by using $\{\hat{x}\}_1 = \frac{1}{(a_1^2 + \ldots + a_n^2)^{1/2}} \begin{Bmatrix} a_1 \\ * \\ * \\ a_n \end{Bmatrix}$ ;

c) Deflate the original coefficient [A] by using $[A_1] = [A] - \lambda_1 \{\hat{x}\}_1 \{\hat{x}\}_1^T$ ;

d) Use the method outlined in section (6-a) to calculate the largest eigenvalue of deflated $[A_1]$ matrix.

e) Repeat steps ( b- d ) until all eigenvalues are determined.

**Home Work : Determine the intermediate Eigenvalues of**

$$[A] = \begin{bmatrix} 3.556 & -1.778 & 0 \\ -1.778 & 3.556 & -1.778 \\ 0 & -1.778 & 3.556 \end{bmatrix}$$

## 7. Transformation methods ( Jacobi Method)

Let Eigen Problems involves symmetric [A] matrices & identity [B] matrices. That is;

$$[[A] - \lambda[I]]\{X\} = \{0\}$$

Jacobi method states that; the transformation of a symmetrical [A] matrix *( i.e. $a_{ij} = a_{ji}$ )* into a diagonal one having the same eigenvalues as the original [A] matrix. Given

$$[A]\ \{x\} = \lambda\ \{x\} \qquad (7\text{-}1)$$

where [A] is a symmetrical matrix ( *n* x *n* );

Suppose that eq. (7-1) produces the eigenvalues $\lambda_1$ , $\lambda_2, \lambda_3, \ldots, \lambda_n$ and their corresponding vectors $\{V_1\}, \{V_2\}, \ldots\ldots, \{V_n\}$; then

$[A]\ \{V_1\} = \lambda_1\ \{V_1\}$

$[A]\ \{V_2\} = \lambda_2\ \{V_2\}$

*

$[A]\ \{V_n\} = \lambda_n \{V_n\}$

In compact form it gives ***[A][V] = [V][λ]*** ( 7-2)

where $[V] = [\{v_1\}, \{v_2\}, \ldots, \{v_n\}]$ ; $[\lambda]=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & * & 0\\ 0 & 0 & \lambda_n\end{bmatrix}$

Since [A] is symmetrical ( *i.e.* $a_{ij} = a_{ji}$ ) , then its eigenvectors are orthogonal, the implication is that $[V]^T = [V]^{-1}$

Premultiplying eq. (7-2) by $[V]^T$ gives $[V]^T[A][V] = [V][\lambda]\,[V]^{-1}$ but $[V]^{-1}\,[V] = [I]$ then;

$$[V]^T[A][V] = \lambda \qquad (7\text{-}3)$$

Then the eigenvalues of [A] are readily determined as the diagonal coefficients of the transformed matrix.

The basic computational scheme is to reduce the original [A] matrix to a diagonal form using real plane rotation matrices as follows;

$[A_1] = [A]$

$[A_2] = [P_1]^T[A_1][P_1]$

$[A_3] = [P_2]^T[A_2][P_2] =[P_2]^T[P_1]^T[A_1][P_1]\,[P_2]$

**********

$[A_k] = [P_{k-1}]^T \ldots.. [P_1]^T\,[A]\,[P_1]\ldots\ldots[P_{k-1}]$

Consequently as $k\rightarrow\rightarrow \infty$ then $[A_k] \rightarrow\rightarrow \lambda$ and $[P_1]\ldots..[P_{k-1}] \rightarrow\rightarrow [V]$

The $[P_k]$ is a plane rotation matrix which is constructed in such a way that an off- diagonal coefficients in $[A_k]$ is reduced to zero. Hence if $a_{rs}$ coefficient is to be reduced to zero, then

$$[P_k]=\begin{array}{c} \\ \\ r \\ \\ s \\ \\ \end{array}\begin{array}{c}\begin{matrix} 1 & 0 & r & 0 & s & 0 & 0\end{matrix}\\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \cos\vartheta & 0 & -\sin\vartheta & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & \sin\vartheta & 0 & \cos\vartheta & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}\end{array} \qquad (7\text{-}4)$$

$[P_k]$ matrix is an orthogonal matrix.

$P_{rr}= P_{ss} = \cos\vartheta$

$P_{rs}= - P_{sr} = \sin\vartheta$

$P_{ii}= 1$ for $i \neq r, s$

$P_{ij}= 0$ elsewhere

The value of $\vartheta$ is selected in such a way that $a_{rs}$ coefficient in $[A_k]$ matrix is reduced to zero. That is

$$\tan 2\vartheta = \frac{2a_{rs}}{a_{rr} - a_{ss}} \qquad (7\text{-}5)$$

**Example (7-1): Prove eq. (7-5) for 2 x 2 symmetrical matrix**

**Solution:**

$$[A] = \begin{array}{c} \\ r \\ s \end{array}\begin{array}{c}\begin{matrix} r & s\end{matrix}\\ \begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\end{array} = [A_1] \quad ; r =1 , s =2$$

using eq. ( 7-4 ) $[P_1] = \begin{bmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{bmatrix}$

$\rightarrow\rightarrow [A_2] = [P_1]^T[A_1][P_1]$

$$\begin{bmatrix} a_{11} & a_{12}\\ a_{12} & a_{22}\end{bmatrix} = \begin{bmatrix}\cos\vartheta & \sin\vartheta\\ -\sin\vartheta & \cos\vartheta\end{bmatrix}\begin{bmatrix} a_{11} & a_{12}\\ a_{12} & a_{22}\end{bmatrix}\begin{bmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{bmatrix}$$ ; since $a_{ij}=a_{ji}$ , $a_{21}$ replaced by $a_{12}$ , our task is to reduce $a_{12}$ to zero

$a_{12}=0=\begin{bmatrix} a_{11}\cos\vartheta + a_{12}\sin\vartheta & a_{12}\cos\vartheta + a_{22}\sin\vartheta \end{bmatrix}\begin{bmatrix} -\sin\vartheta \\ \cos\vartheta \end{bmatrix}$

$a_{12}=0=a_{11}\cos\vartheta\,(-\sin\vartheta) + a_{12}\sin\vartheta\,(-\sin\vartheta) + a_{12}(\cos\vartheta)^2 + a_{22}\sin\vartheta\cos\vartheta$

$a_{12}=0 = -\sin\vartheta\,(\cos\vartheta)a_{11} + (\cos\vartheta)^2 a_{12} - a_{12}(\sin\vartheta)^2 + \cos\vartheta(\sin\vartheta)\,a_{12}$

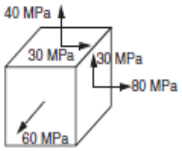
Simplifying $\sin\vartheta(\cos\vartheta)a_{11} - \sin\vartheta(\cos\vartheta)a_{22} = a_{12}((\cos\vartheta)^2 - (\sin\vartheta)^2)$

and rearranging gives;

$$\frac{\sin\vartheta\ \cos\vartheta}{(\cos\vartheta)^2-(\sin\vartheta)^2} = \frac{a_{12}}{a_{11}-a_{22}} \longrightarrow \frac{\frac{1}{2}\sin 2\vartheta}{\cos 2\vartheta} = \frac{a_{12}}{a_{11}-a_{22}}$$

or $\tan 2\vartheta = \frac{2}{1}\left[\frac{a_{12}}{a_{11}-a_{22}}\right] = \frac{2a_{12}}{a_{11}-a_{22}}$ ; That is mean 2 x 2 matrix requires a single iteration for $[A_1]$ to becomes a diagonal one.

**Example (7-2):**




$$[S]=\begin{bmatrix} 80 & 30 & 0 \\ 30 & 40 & 0 \\ 0 & 0 & 60 \end{bmatrix}\text{MPa}$$

The stress matrix ( tensor ) corresponding to the state of the above Fig. is shown in [S] matrix. Determine ***EigenValues*** **( the principal stresses)** & ***the associated EigenVectors*** of the state by ***Jacobi method.***

**Solution:**

Since the larger element in the upper triangle of the [ S ] matrix is $a_{12}=30$ then take r=1 , s=2; hence

$\tan 2\vartheta = \frac{2a_{12}}{a_{11}-a_{22}} = \frac{2(30)}{80-40} = \frac{3}{2} \longrightarrow 2\vartheta = 56.31 \longrightarrow \vartheta = 28.155$

$$P=\begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.8817 & -0.4719 & 0 \\ 0.4719 & 0.8817 & 0 \\ 0 & 0 & 1 \end{bmatrix};\ P^T=\begin{bmatrix} 0.8817 & 0.4719 & 0 \\ -0.4719 & 0.8817 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_0=[S]=\begin{bmatrix} 80 & 30 & 0 \\ 30 & 40 & 0 \\ 0 & 0 & 60 \end{bmatrix}$$

then $A_1 = P^T \cdot A_0 \cdot P = \begin{bmatrix} 0.8817 & 0.4719 & 0 \\ -0.4719 & 0.8817 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 80 & 30 & 0 \\ 30 & 40 & 0 \\ 0 & 0 & 60 \end{bmatrix}\begin{bmatrix} 0.8817 & -0.4719 & 0 \\ 0.4719 & 0.8817 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A_1 = \begin{bmatrix} 96.064 & 0 & 0 \\ 0 & 23.946 & 0 \\ 0 & 0 & 60 \end{bmatrix}$$ ; hence $\{\lambda_i\} = \{96.064, 23.946, 60\}$

and EigenVector = $V=\begin{bmatrix} 0.8817 & -0.4719 & 0 \\ 0.4719 & 0.8817 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. **Householder's Method** ( Al-Khafaji 257)

This solution technique permits the transformation of a symmetrical matrix into a tridiagonal form having the same eigenvalues. It uses a more complex transformation than the Jacobi method to reduce whole rows and columns of off-diagonal coefficients to zero. However, it is significantly more efficient than the Jacobi technique in that it required a finite number of iterations. The diagonal form can then be used to construct a sequence of polynomials that can be solved for the eigenvalues. In fact, the Jacobi as well as other techniques can be used to calculate the eigenvalues, once the

tridiagonal form achieved. The Householder method requires the construction of reflection matrices and n-2 transformations of the form.

$[A_1] = [A]$

$[A_2] = [P_1]^T[A_1][P_1]$

**********

$[A_{n-2}] = [P_{n-3}]^T[A_{n-3}][P_{n-3}]$

where $n$ is the order of the [A] matrix under consideration. These equations can be summarized more simply as follws:

$[A_{kh}] = [P_k]^T[A_k][P_k]$ , $k= 1, \dots ,n-2$ *(8-1)*

The $[P_k]$ matrices are the Householder transformation matrices and can be constructed from the following:

$$[P_k] = [I]\text{-}S_k \{W_k\}\{W_k\}^T \quad (8\text{-}2a)$$

$$S_k = \frac{2}{\{W_k\}^T\{W_k\}} \quad (8\text{-}2b)$$

The coefficients of the $\{W_k\}$ vector are defined in terms of the [A] matrix coefficients as follows:

$$W_{ik} = \begin{cases} 0 & for\ i = 1, 2, \dots, k \\ a_{ik} & for\ i = k+2, \dots, n \end{cases}$$

$$W_{k+1,k} = a_{k+1,k} \mp \sqrt{\sum_{i=k+1}^{n} a_{ik}^2}$$

The sign preceding the square root is taken to be the same as that for the coefficient $a_{k+1,k}$ . Consequently, once a tridiagonal form has been achieved, it then becomes necessary to calculate the eigenvalues. This is accomplished by expanding the following determinant:

$$\det[T] = \begin{vmatrix} A_1 - \lambda & B_2 & & & \\ B_2 & A_2 - \lambda & B_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & B_n \\ & & & B_n & A_n - \lambda \end{vmatrix} = 0$$

where $[T]$ is the tridiagonal form of the original [A] matrix. The determinant $\det[T]$ is given by the following polynomials:

$f_0(\lambda) = 1$

$f_1(\lambda) = A_1 - \lambda$

$f_m(\lambda) = (A_m - \lambda) f_{m-1}(\lambda) - B_m^2 f_{m-2}(\lambda)$ *(8-3)*

That is, using the quadratic formula ( *m-2)*, one is able to identify approximations for two of the roots, therefore, for *m=3* a third root is approximated, etc. This information is quite valuable when using the Newton- Raphson methods for root approximations of nonlinear algebraic equations.

- **Householder's Method Algorithm**

for K[th] iteration;

1. find $\{W_k\} \rightarrow W_{ik} = \begin{cases} 0 & for\ i = 1, 2, \dots, k \\ a_{ik} & for\ i = k+2, \dots, n \end{cases}$ ; $W_{k+1,k} = a_{k+1,k} \mp \sqrt{\sum_{i=k+1}^{n} a_{ik}^2}$

   the sign ± is to be taken as + when $a_{k+1,k}$ has +ve sign in $A_1$ matrix; otherwise it is to be taken – ve sign.

2. find $S_k = \frac{2}{\{W_k\}^T\{W_k\}}$

3. Reflect $[P_k]$ householder Transformation Matrix→$[P_k] = [I]\text{-}S_k \{W_k\}\{W_k\}^T$
4. Calculate the Tridiagonal Matrix $[A_{k+1}]$ → $[A_{kh}] = [P_k]^T[A_k][P_k]$
5. If $[A_{kh}]$ is not Tridiagonal then put k=k+1 and repeat steps 1 to 4, else goto step 6.
6. Calculate EigenValues → $f_m(\lambda)=(A_m - \lambda)f_{m-1}(\lambda) - B_m^2 f_{m-2}(\lambda)$

**Example (8-1)** use the algorithm to find the eigenvalues of the following matrix;

$$[A_1] = \begin{bmatrix} 30 & 6 & 5 \\ 6 & 30 & 9 \\ 5 & 9 & 30 \end{bmatrix}$$

**Solution:**

let k=1

1. $W_{11}$ =0
   $W_{31} = a_{31}$=5
   $W_{21}=a_{21}\pm\sqrt{a_{21}^2 + a_{31}^2}$ = 6 + $\sqrt{6^2 + 5^2}$ = 13.81
   $\{w_1\} = \begin{Bmatrix} 0 \\ 13.81 \\ 5 \end{Bmatrix}$
2. $S_1 = \frac{2}{0^2+13.81^2+5^2}$ = 0.0093
3. $[P_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 0.0093 \begin{Bmatrix} 0 \\ 13.81 \\ 5 \end{Bmatrix} \{0 \quad 13.81 \quad 5\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.7682 & -0.6402 \\ 0 & -0.6402 & 0.7682 \end{bmatrix}$
4. $[A_2] = [P_1]^T[A_1][P_1] = \begin{bmatrix} 30 & -7.81 & 0 \\ -7.81 & 38.85 & -1.622 \\ 0 & -1.622 & 21.15 \end{bmatrix}$
5. $[A_2]$ is Tridiagonal Matrix; then goto step 6
6. $f_2(\lambda)=(A_2 - \lambda)f_1(\lambda) - B_2^2 f_0(\lambda) = (38.85 - \lambda)(30 - \lambda) - (-7.81)^2$ (1)
   $= \lambda^2 - 68.85\,\lambda + 1104.50$

The roots are given as $\lambda_1$= 43.4 , $\lambda_2$= 25.45;

for m=3

$$f_3(\lambda) = (A_3 - \lambda)f_2(\lambda) - (-1.622)^2 f_1(\lambda)$$

$$= (21.15 - \lambda)(\lambda^2 - 68.85\lambda + 1104.50) - 2.63(30 - \lambda)$$

the 3rd root is $\lambda_3$= 21.15; is better approximation to all of the roots, with starting roots $\lambda_1$ , $\lambda_2$ , $\lambda_3$.

# Lecture 5: INTEGRATION

**What is integration?**

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to do so? Among the most common examples are finding the velocity of a body from acceleration functions, and displacement of a body from velocity data. Throughout the engineering fields, there are (what some times seems like) countless applications for integral calculus. Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods have been developed to find the integral.

Here, we will discuss the trapezoidal rule of approximating integrals of the form

$$I = \int_a^b f(x)dx$$

$f(x)$ is called the integrand,
$a$ = lower limit of integration
$b$ = upper limit of integration

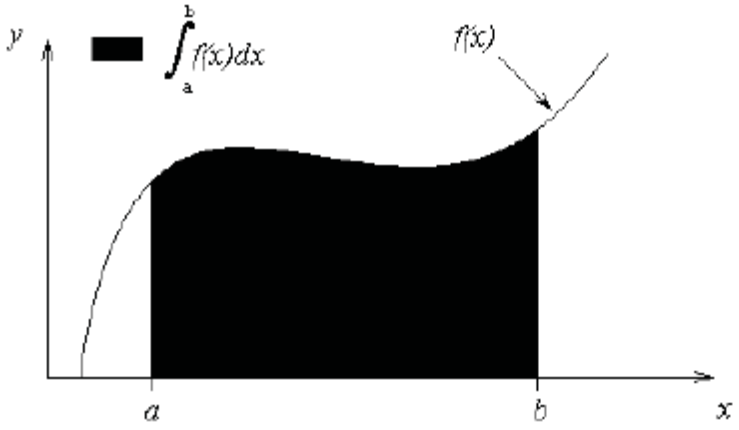



**Figure 1: Integration of a function**

## TRAPEZOIDAL RULE:

Trapezoidal rule is based on the Newton-Cotes formula that if one approximates the integrand by an n[th] order polynomial, then the integral of the function is approximated by the integral of that n[th] order polynomial. Integrating polynomials is simple and is based on the calculus formula.

$$\int_a^b x^n dx = \left( \frac{b^{n+1} - a^{n+1}}{n+1} \right), \; n \geq 0 \qquad (1$$

So if we want to approximate the integral

$$I = \int_a^b f(x)dx \qquad (2$$

to find the value of the above integral, one assumes

$$f(x) \approx f_n(x) \qquad (3$$

where

$$f_n(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + a_n x^n . \qquad (4$$

where $f_n(x)$ is an $n^{th}$ order polynomial. Trapezoidal rule assumes $n$ =1, that is, the area under the linear polynomial (straight line),

$$\int_a^b f(x)dx \approx \int_a^b f_1(x)dx$$

## DEREVATION OF THE TRAPEZOIDAL RULE:

### Method 1: Derived from Calculus

Hence

$$\int_a^b f(x)dx \approx \int_a^b f_1(x)dx$$

$$= \int_a^b (a_0 + a_1 x)dx$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right). \quad (5$$

But what is $a_0$ and $a_1$? Now if one chooses, *(a, f(a))* and *(b, f(b))* as the two points to approximate by a straight line from *a* to *b*,

$$f(a) = f_1(a) = a_0 + a_1 a \quad (6$$

$$f(b) = f_1(b) = a_0 + a_1 b \quad (7$$

Solving the above two equations for *a* and *b*,

$$a_1 = \frac{f(b) - f(a)}{b - a}$$

$$a_0 = \frac{f(a)b - f(b)a}{b - a} \quad (8$$

Hence from Equation (5),

$$\int_a^b f(x)dx = \frac{f(a)b - f(b)a}{b-a}(b-a) + \frac{f(b) - f(a)}{b-a}\frac{b^2 - a^2}{2}$$

$$= (b-a)\left[\frac{f(a) + f(b)}{2}\right] \quad (9$$

### Method 2: Derived from Geometry

The Trapezoidal rule can also be derived from geometry. Look at Figure 2. The area under the curve $f_1(x)$ is a trapezoid. The integral $\int_a^b f(x)dx \approx Area\ of\ trapezoid$

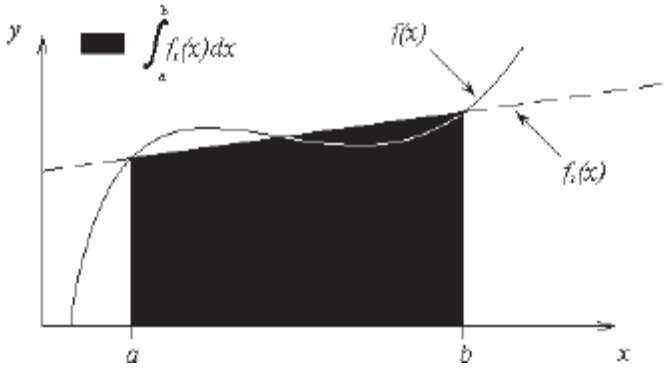



**Figure 2: Geometric Representation**

=(1/2)( *Sum of parallel sides)(height)*