$$= \frac{1}{2}\left(f(b) + f(a)\right)(b-a)$$

$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right] \quad (10$$

Method 3: Derived from Method of Coefficients

Trapezoidal rule can also be derived by the method of coefficients. The formula

$$\int_a^b f(x)dx \approx \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)$$

$$= \sum_{i=1}^{2} c_i f(x_i) \quad (11$$

where $c_1=(b-a)/2$; $c_2=(b-a)/2$; $x_1=a$; and $x_2=b$

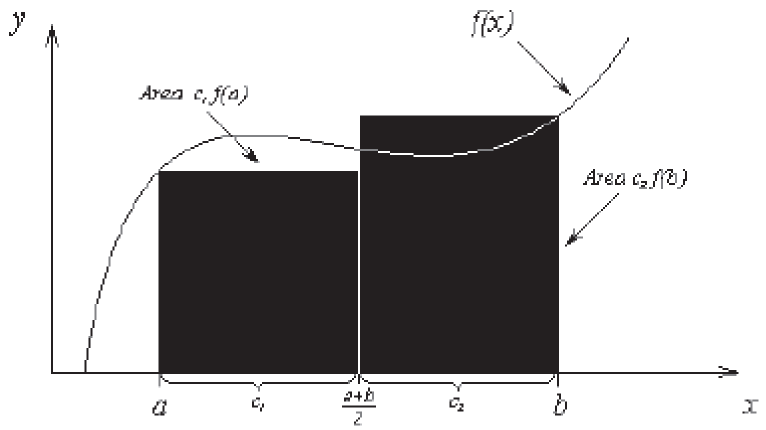


Figure 3: Area by Coefficients

The interpretation is that $f(x)$ is evaluated at points a and b, and each function evaluation is given a weight of $(b-a)/2$. Geometrically, Equation (10) is looked at as an area of a trapezoid, while Equation (11) is viewed as the sum of the area of two rectangles, as shown in Figure 3. How can one derive trapezoidal rule by the method of coefficients?

Assume

$$\int_a^b f(x)dx = c_1 f(a) + c_2 f(b) \quad (12$$

Let the right hand side be an exact expression for integrals of $\int_a^b 1dx$ and $\int_a^b x\,dx$, that is, the formula will then also be exact for linear combinations of $f(x) = 1$ and $f(x) = x$, that is, $f(x) = a_0\,(1) + a_1\,(x)$.

$$\int_a^b 1dx = b - a = c_1 + c_2 \quad (13$$

$$\int_a^b xdx = \frac{b^2-a^2}{2} = c_1 a + c_2 b \quad (14$$

Solving the above two equations gives

$$c_1 = \frac{b-a}{2} \quad ; \quad c_2 = \frac{b-a}{2} \quad (15$$

Hence

$$\int_a^b f(x)dx \approx \frac{b-a}{2}f(a) + \frac{b-a}{2}f(b) \qquad (16$$

Multiple-segment Trapezoidal Rule:

Divide (*b – a*) into *n* equal segments as shown in Figure 4. Then the width of each segment is *h=(b – a)/n*

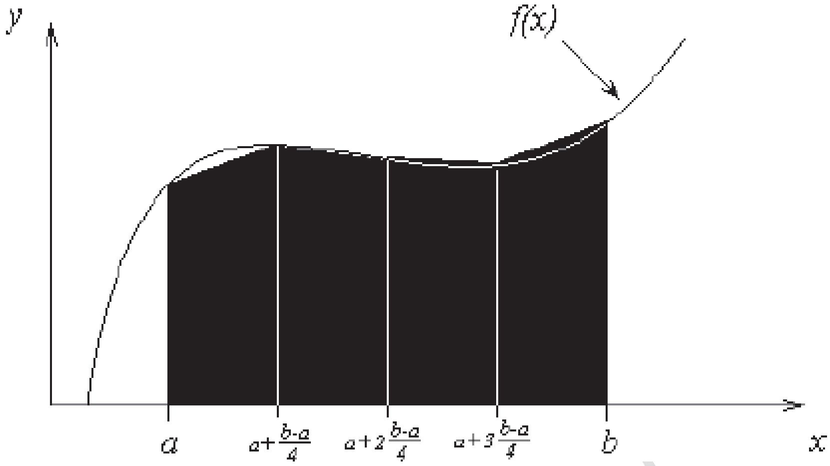


Figure (4) : Multiple (n=4) Segment Trapezoidal Rule

The integral *I* can be broken into *h* integrals as

$$I = \int_a^b f(x)dx$$

$$= \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \ldots + \int_{a+(n-2)h}^{a+(n-1)h} f(x)dx + \int_{a+(n-1)h}^{b} f(x)dx \qquad (17$$

Applying Trapezoidal rule Equation (17) on each segment gives

$$\int_a^b f(x)dx = [(a+h)-a]\left[\frac{f(a)+f(a+h)}{2}\right]$$

$$+[(a+2h)-(a+h)]\left[\frac{f(a+h)+f(a+2h)}{2}\right]$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$

$$+[(a+(n-1)h)-(a+(n-2)h)]\left[\frac{f(a+(n-2)h)+f(a+(n-1)h)}{2}\right]$$

$$+[b-(a+(n-1)h)]\left[\frac{f(a+(n-1)h)+f(b)}{2}\right]$$

$$= h\left[\frac{f(a)+f(a+h)}{2}\right]$$

$$+ h\left[\frac{f(a+h)+f(a+2h)}{2}\right]$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$

$$+h\left[\frac{f(a+(n-2)h)+f(a+(n-1)h)}{2}\right]$$

$$+h\left[\frac{f(a+(n-1)h)+f(b)}{2}\right]$$

$$=h\left[\frac{f(a)+2f(a+h)+2f(a+2h)+\ldots+2f(a+(n-1)h)+f(b)}{2}\right]$$

$$=\frac{h}{2}\left[f(a)+2\left\{\sum_{i=1}^{n-1}f(a+ih)\right\}+f(b)\right]$$

$$=\frac{b-a}{2n}\left[f(a)+2\left\{\sum_{i=1}^{n-1}f(a+ih)\right\}+f(b)\right] \quad (18$$

Example 1

The vertical distance covered by a rocket from *t=8 to t=30* seconds is given by

$$x=\int_{8}^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

a) Use single segment Trapezoidal rule to find the distance covered.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error for part (a).

Solution

a) $$I\approx(b-a)\left[\frac{f(a)+f(b)}{2}\right], \text{ where}$$

$$a=8$$
$$b=30$$

$$f(t)=2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t$$

$$f(8)=2000\ln\left[\frac{140000}{140000-2100(8)}\right]-9.8(8)$$
$$=177.27\ m/s$$

$$f(30)=2000\ln\left[\frac{140000}{140000-2100(30)}\right]-9.8(30)$$
$$=901.67\ m/s$$

$$I=(30-8)\left[\frac{177.27+901.67}{2}\right]$$
$$=11868\ m$$

b) The exact value of the above integral is

$$x=\int_{8}^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$
$$=11061\ m$$

so the true error is

$$E_t = True\ Value - Approximate\ Value$$
$$= 11061 - 11868$$
$$= -807\ m$$

c) The absolute relative true error, $|\epsilon_t|$, would then be

$$|\epsilon_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100$$
$$= \left|\frac{11061 - 11868}{11061}\right| \times 100$$
$$= 7.2959\%$$

Example 2

The vertical distance covered by a rocket from *t=8 to t=30* seconds is given by

$$x = \int_8^{30} \left(2000 \ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t \right) dt$$

a) Use two-segment Trapezoidal rule to find the distance covered.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error for part (a).

Solution:

a) The solution using 2-segment Trapezoidal rule is

$$I = \frac{b-a}{2n}\left[f(a) + 2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\} + f(b)\right]$$

n=2; a=8; b=30; h=(b – a)/n=(30 – 8)/2 =11

$$I = \frac{30-8}{2(2)}\left[f(8) + 2\left\{\sum_{i=1}^{2-1} f(a+ih)\right\} + f(30)\right]$$
$$= \frac{22}{4}[f(8) + 2f(19) + f(30)]$$
$$= \frac{22}{4}[177.27 + 2(484.75) + 901.67]$$
$$= 11266\ m$$

b) The exact value of the above integral is

$$x = \int_8^{30} \left(2000 \ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t \right) dt$$
$$= 11061\ m$$

so the true error is

$$E_t = True\ Value - Approximate\ Value$$
$$= 11061 - 11266$$
$$= -205\ m$$

c) The absolute relative true error, $|\epsilon_t|$, would then be

$$
\begin{aligned}
|\in_t| &= \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100 \\
&= \left|\frac{11061-11266}{11061}\right| \times 100 \\
&= 1.8534\%
\end{aligned}
$$

Table 1: Values obtained using multiple-segment Trapezoidal rule for

$$x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

n	Value	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
1	11868	-807	7.296	---
2	11266	-205	1.853	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482
8	11074	-12.9	0.1165	0.03560

<u>**Home Works**</u>

1. Use Multiple-segment Trapezoidal Rule to find the area under the curve

$$f(x) = \frac{300x}{1+e^x}$$

from $x=0$ to $x=10$.

Table 2: Values obtained using Multiple-segment Trapezoidal Rule for $\int_0^{10}\frac{300x}{1+e^x}dx$

n	Approximate Value	E_t	$\lvert\in_t\rvert$
1	0.681	245.91	99.724%
2	50.535	196.05	79.505%
4	170.61	75.978	30.812%
8	227.04	19.546	7.927%
16	241.70	4.887	1.982%
32	245.37	1.222	0.495%
64	246.28	0.305	0.124%

2. Use multiple-segment Trapezoidal Rule to find

$$I = \int_0^2 \frac{1}{\sqrt{x}}\,dx$$

Table 3: *Values obtained using Multiple-segment Trapezoidal Rule for* $\int_0^2 \frac{1}{\sqrt{x}}dx$

Number of Segments	Approximate Value	E_t	$\lvert\epsilon_t\rvert$
2	1.354	1.474	52.14%
4	1.792	1.036	36.64%
8	2.097	0.731	25.85%
16	2.312	0.516	18.26%
32	2.463	0.365	12.91%
64	2.570	0.258	9.128%
128	2.646	0.182	6.454%
256	2.699	0.129	4.564%
512	2.737	0.091	3.227%
1024	2.764	0.064	2.282%
2048	2.783	0.045	1.613%
4096	2.796	0.032	1.141%

SIMPSON'S 1/3^RD RULE:

Trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial in the interval of integration. Simpson's 1/3rd rule is an extension of Trapezoidal rule where the integrand is non approximated by a second order polynomial.

Method 1: Hence

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

where $f_2(x)$ is a second order polynomial. $f_2(x) = a_0 + a_1x + a_2x^2$

Choose $(a, f(a)), ((a+b)/2, f((a+b)/2))$ and $(b, f(b))$ as the three points of the function to evaluate a_0, a_1, and a_2.

$$f(a) = f_2(a) = a_0 + a_1a + a_2a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1b + a_2b^2$$

Solving the above three equations for unknowns, a_0, a_1 and a_2 give

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Then

$$I \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left(a_0 + a_1 x + a_2 x^2\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_a^b$$

$$= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3}$$

Substituting values of a_0 , a_1 and a_2 give

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since for Simpson's 1/3rd Rule, the interval $[a, b]$ is broken into 2 segments, the segment width $h=(b-a)/2$

Hence the Simpson's 1/3rd rule is given by

$$\boxed{\int_a^b f(x)dx \cong \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]}$$

Method 2:

Simpson's formula could also be derived by approximating *f(x)* by a second order polynomial using Newton's divided difference polynomial as

$$f_2(x) = b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right)$$

where

$$b_0 = f(a)$$

$$b_1 = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}$$

$$b_2 = \frac{\dfrac{f(b) - f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}} - \dfrac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}}{b-a}$$

Integrating Newton's divided difference polynomial gives us

$$\int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left[b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right)\right]dx$$

$$= \left[b_0 x + b_1\left(\frac{x^2}{2} - ax\right) + b_2\left(\frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2}\right)\right]_a^b$$

$$= b_0(b-a) + b_1\left(\frac{b^2-a^2}{2} - a(b-a)\right) + b_2\left(\frac{b^3-a^3}{3} - \frac{(3a+b)(b^2-a^2)}{4} + \frac{a(a+b)(b-a)}{2}\right)$$

Substituting values of b_0 , b_1 , and b_2 into this equation yields the same result as before

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$\boxed{\int_a^b f(x)dx \cong \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]}$$

Method 3:

Simpson's 1/3rd Rule can also be derived by the method of coefficients. Assume

$$\int_a^b f(x)dx \cong c_1 f(a) + c_2 f\left(\frac{a+b}{2}\right) + c_3 f(b)$$

Let the right-hand side be an exact expression for integrals $\int_a^b 1dx,\ \int_a^b xdx,\ \text{and}\ \int_a^b x^2dx$. Doing this will imply that the right hand side will be exact expressions for integrals of any linear combination of the three integrals, implying it for a general second order polynomial. Now

$$\int_a^b 1dx = b - a = c_1 + c_2 + c_3$$

$$\int_a^b xdx = \frac{b^2-a^2}{2} = c_1 a + c_2\frac{a+b}{2} + c_3 b$$

$$\int_a^b x^2dx = \frac{b^3-a^3}{3} = c_1 a^2 + c_2\left(\frac{a+b}{2}\right)^2 + c_3 b^2$$

Solving the above 3 equations for c_o , c_1 and c_2 give

$$c_1 = \frac{b-a}{6} \quad ; \quad c_2 = \frac{2(b-a)}{3} \quad ; \quad c_3 = \frac{b-a}{6}$$

This gives

$$\int_a^b f(x)dx = \frac{b-a}{6}f(a) + \frac{2(b-a)}{3}f\left(\frac{a+b}{2}\right) + \frac{b-a}{6}f(b)$$

$$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

The integral from the first method,

$$\int_a^b f(x)dx = \int_a^b \left(a_0 + a_1x + a_2x^2\right)dx$$

can be viewed as the area under the second order polynomial, while the equation from this past method

$$\int_a^b f(x)dx = \frac{b-a}{6}f(a) + \frac{2(b-a)}{3}f\left(\frac{a+b}{2}\right) + \frac{b-a}{6}f(b)$$

can be viewed as the sum of the areas of three rectangles.

Multiple Segment Simpson's 1/3rd Rule

Just like in multiple-segment Trapezoidal Rule, one can subdivide the interval [*a* , *b*] into *n* segments and apply Simpson's 1/3rd Rule repeatedly over every two segments. Note that *n* needs to be even. Divide interval [*a* , *b*] into equal segments, hence the segment width *h= (b –a)/n* .

$$\int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx$$

where

$x_0 = a$

$x_n = b$

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots\ldots + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$

Apply Simpson's 1/3rd Rule over each interval,

$$\int_a^b f(x)dx \cong (x_2 - x_0)\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right] + (x_4 - x_2)\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right] + \ldots$$
$$+ (x_{n-2} - x_{n-4})\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right] + (x_n - x_{n-2})\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

Since

$$x_i - x_{i-2} = 2h$$
$$i = 2, 4, \ldots, n$$

then

$$\int_a^b f(x)dx \cong 2h\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right] + 2h\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right] + \ldots$$
$$+ 2h\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right] + 2h\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$
$$= \frac{h}{3}\left[f(x_0) + 4\{f(x_1)+f(x_3)+\ldots+f(x_{n-1})\} + 2\{f(x_2)+f(x_4)+\ldots+f(x_{n-2})\} + f(x_n)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n)\right]$$

$$\boxed{\int_a^b f(x)dx \cong \frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n)\right]}$$

GAUSSIAN- QUADRATURE FORMULAS

We wish to find the area under the curve $y = f(x)$, $-1 \le x \le 1$.
What method gives the best answer if only two function evaluations are to be made?
We have already seen that the trapezoidal rule is a method for finding the area under the curve and that it uses two function evaluations at the end points $(-1, f(-1))$, and $(1, f(1))$. But if the graph of $y = f(x)$ is concave down, the error in approximation is the entire region that lies between the curve and the line segment joining the points; another instance is shown in Figure 7.10(a).

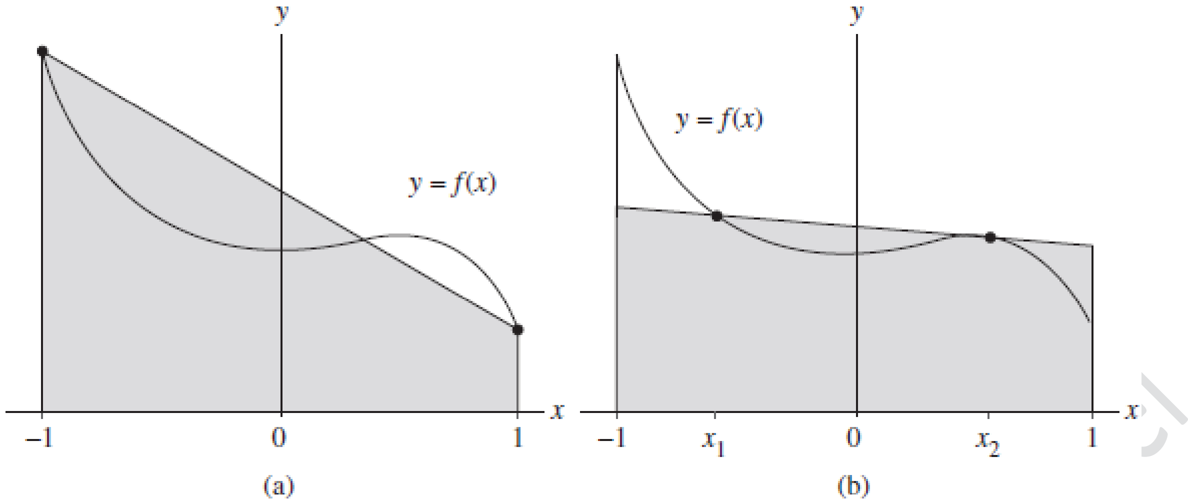


Figure 7.10 (a) Trapezoidal approximation using the abscissas −1 and 1. (b) Trapezoidal approximation using the abscissas x_1 and x_2.

If we can use nodes $x1$ and $x2$ that lie inside the interval [−1, 1], the line through the two points $(x1, f(x1))$ and $(x2, f(x2))$ crosses the curve, and the area under the line more closely approximates the area under the curve (see Figure 7.10(b)). The equation of the line is

$$y = f(x_1) + \frac{(x - x_1)(f(x_2) - f(x_1))}{x_2 - x_1} \qquad (1$$

and the area of the trapezoid under the line is

$$A_{\text{trap}} = \frac{2x_2}{x_2 - x_1} f(x_1) - \frac{2x_1}{x_2 - x_1} f(x_2). \qquad (2$$

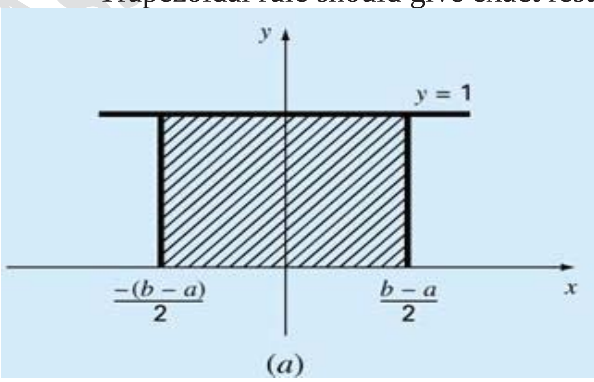
Notice that the trapezoidal rule is a special case of (2). When we choose $x1 = -1$, $x2 = 1$, and $h = 2$, then

$$T(f, h) = \frac{2}{2} f(x_1) - \frac{-2}{2} f(x_2) = f(x_1) + f(x_2).$$

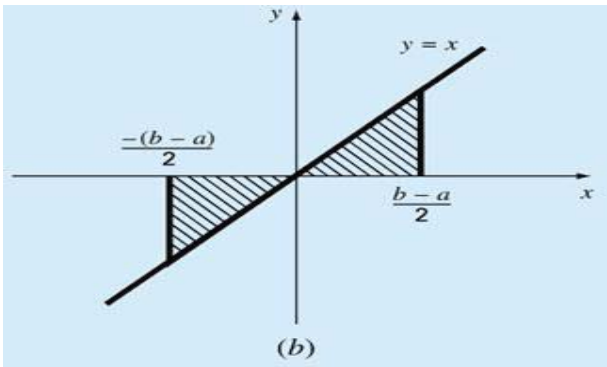
Gauss Quadrature

$$\boxed{I = \int_a^b f(x)dx}$$

- Assume $\boxed{I \cong c_0 f(a) + c_1 f(b)}$
- a and b are limits of integration
- Trapezoidal rule should give exact results for constant and linear functions



Constant Function



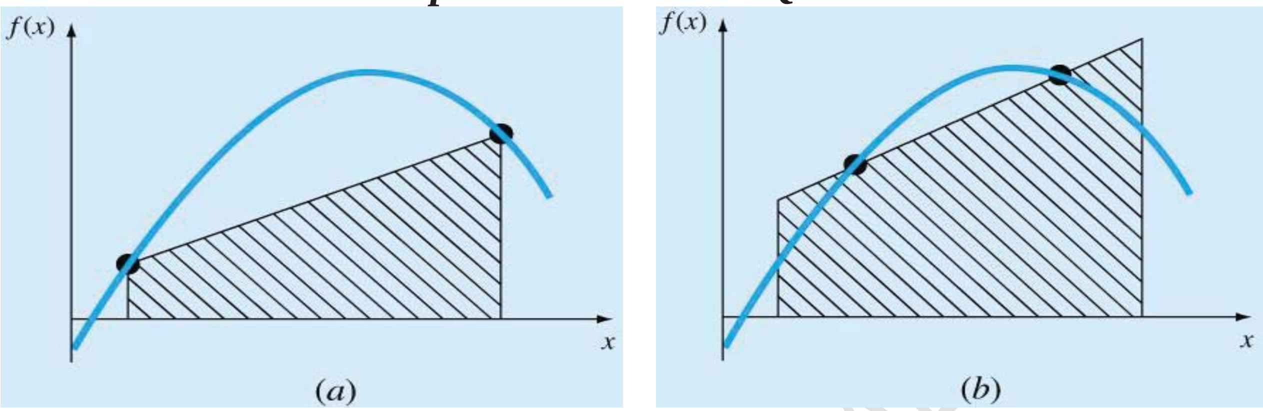
Linear Function

- Now instead of trapezoidal, which has fixed end points (a,b), let them float
- 4 unknowns - x_0, x_1, c_0, c_1

$$\boxed{I = \int_{-1}^{1} f(x)dx = c_0 f(x_0) + c_1 f(x_1)}$$

- 4 equations - constant, linear (had before in trapezoidal rule), quadratic, cubic
- Integrate from -1 to 1 to simplify math

Trapezoidal vs. Gauss-Quadrature

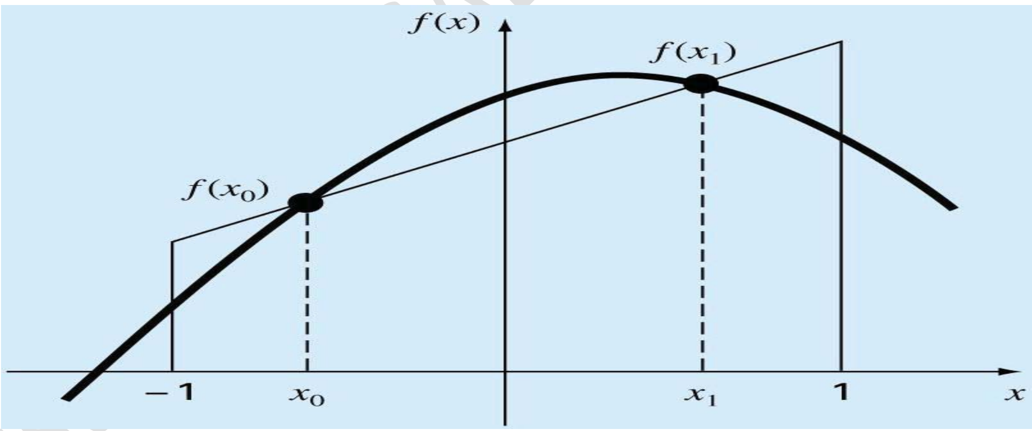


Exact for constant and linear function functions

Exact for constant, linear, quadratic and cubic

- The idea is that if we evaluate the function at certain points (non-uniformly distributed), and sum with certain weights, we will get accurate integral
- Evaluation points and weights are tabulated

Gauss-Legendre Quadrature



- Choose (c_0, c_1, x_0, x_1) to yield highest possible accuracy
- change of variables so that the interval of integration is $[-1,1]$
- select functional values at **non-uniformly** distributed points to achieve higher accuracy

Gauss Quadrature on [a, b]

- To go to $[-1,1]$ from other limits $[a,b]$ - use linear transformation
- Change from $a \le x \le b$ to $-1 \le x_d \le 1$

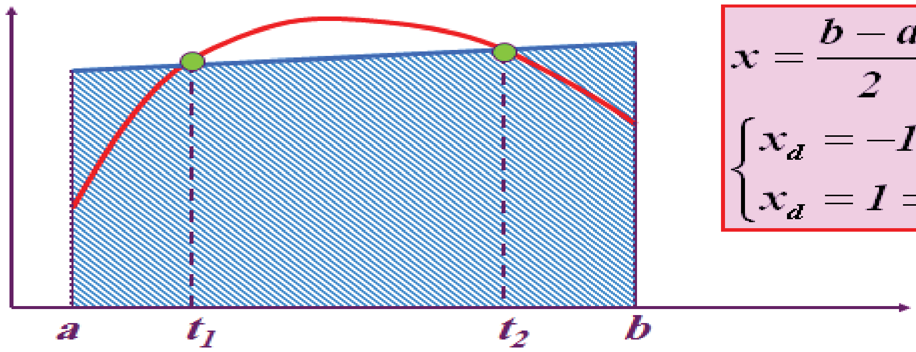
$$x = a_0 + a_1 x_d$$

$$\begin{cases} a = a_0 + a_1(-1) \\ b = a_0 + a_1(1) \end{cases} \Rightarrow \begin{cases} a_0 = (a+b)/2 \\ a_1 = (b-a)/2 \end{cases}$$

- Coordinate transformation

$$\boxed{x = \frac{a+b}{2} + \frac{b-a}{2} x_d}$$

- **Coordinate transformation from [*a*,*b*] to [–1,1]**



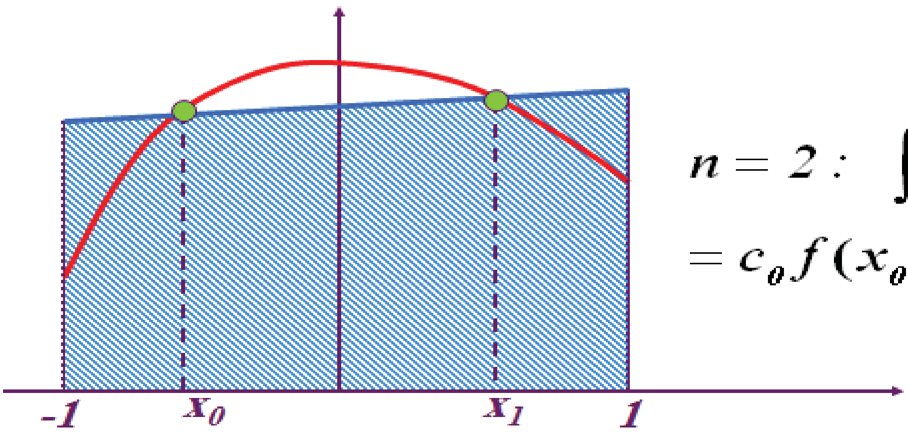
$$x = \frac{b-a}{2}x_d + \frac{a+b}{2}$$

$$\begin{cases} x_d = -1 \Rightarrow x = a \\ x_d = 1 \Rightarrow \quad x = b \end{cases}$$

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}x_d + \frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)dx_d = \int_{-1}^{1} g(x_d)dx_d$$

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i) = c_0 f(x_0) + c_1 f(x_1) + \cdots + c_n f(x_n)$$

<u>For 2 points</u>



$$n = 2: \int_{-1}^{1} f(x)dx = c_0 f(x_0) + c_1 f(x_1)$$

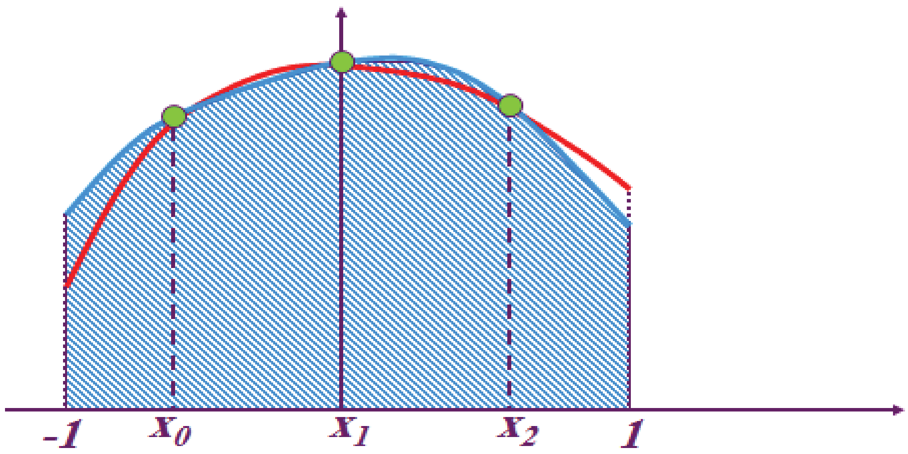
- ***Choose*** **(c_0, c_1, x_0, x_1) such that the method yields "exact integral" for** $f(x) = x^0, x^1, x^2, x^3$
- **Four equations for four unknowns** <u>**(see Eqns 3 through 12 in the next pages)**</u>

$$\begin{cases} f = 1 \Rightarrow \int_{-1}^{1} 1dx = 2 = c_0 + c_1 \\ f = x \Rightarrow \int_{-1}^{1} xdx = 0 = c_0 x_0 + c_1 x_1 \\ f = x^2 \Rightarrow \int_{-1}^{1} x^2 dx = \frac{2}{3} = c_0 x_0^2 + c_1 x_1^2 \\ f = x^3 \Rightarrow \int_{-1}^{1} x^3 dx = 0 = c_0 x_0^3 + c_1 x_1^3 \end{cases} \Rightarrow \begin{cases} c_0 = 1 \\ c_1 = 1 \\ x_0 = \frac{-1}{\sqrt{3}} \\ x_1 = \frac{1}{\sqrt{3}} \end{cases}$$

$$I = \int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

======== For 3 points

$$n = 3: \int_{-1}^{1} f(x)dx = c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2)$$



- *Choose* (c_0, c_1, c_2, x_0, x_1, x_2) **such that the method yields**
- **Exact integral for** $f = x^0, x^1, x^2, x^3, x^4, x^5$

$$\begin{cases} f = 1 \Rightarrow \int_{-1}^{1} 1dx = 2 = c_0 + c_1 + c_2 \\ f = x \Rightarrow \int_{-1}^{1} xdx = 0 = c_0x_0 + c_1x_1 + c_2x_2 \\ f = x^2 \Rightarrow \int_{-1}^{1} x^2dx = \frac{2}{3} = c_0x_0^2 + c_1x_1^2 + c_2x_2^2 \\ f = x^3 \Rightarrow \int_{-1}^{1} x^3dx = 0 = c_0x_0^3 + c_1x_1^3 + c_2x_2^3 \\ f = x^4 \Rightarrow \int_{-1}^{1} x^4dx = 0 = c_0x_0^4 + c_1x_1^4 + c_2x_2^4 \\ f = x^5 \Rightarrow \int_{-1}^{1} x^5dx = 0 = c_0x_0^5 + c_1x_1^5 + c_2x_2^5 \end{cases} \Rightarrow \begin{cases} c_0 = 5/9 \\ c_1 = 8/9 \\ c_2 = 5/9 \\ x_0 = -\sqrt{3/5} \\ x_1 = 0 \\ x_2 = \sqrt{3/5} \end{cases}$$

$$I = \int_{-1}^{1} f(x)dx = \frac{5}{9} f(-\sqrt{\frac{3}{5}}) + \frac{8}{9} f(0) + \frac{5}{9} f(\sqrt{\frac{3}{5}})$$

We shall use the method of undetermined coefficients to find the abscissas x_1, x_2 and weights w_1, w_2 so that the formula

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f(x_1) + w_2 f(x_2) \qquad (3$$

is exact for cubic polynomials (i.e., $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$). Since four coefficients w_1, w_2, x_1, and x_2 need to be determined in equation (3), we can select four conditions to be satisfied. Using the fact that integration is additive, it will suffice to require that (3) be exact for the four functions $f(x) = 1, x, x^2, x^3$. The four integral conditions are

$$\begin{aligned} f(x) = 1: &\quad \int_{-1}^{1} 1\,dx = 2 = w_1 + w_2 \\ f(x) = x: &\quad \int_{-1}^{1} x\,dx = 0 = w_1x_1 + w_2x_2 \\ f(x) = x^2: &\quad \int_{-1}^{1} x^2\,dx = \frac{2}{3} = w_1x_1^2 + w_2x_2^2 \\ f(x) = x^3: &\quad \int_{-1}^{1} x^3\,dx = 0 = w_1x_1^3 + w_2x_2^3. \end{aligned} \qquad (4$$