Now solve the system of nonlinear equations

$w_1 + w_2 = 2$ (5 ; $w_1x_1 = -w_2x_2$ (6 ; $w_1x_1^2 + w_2x_2^2 = 2/3$ (7 ; $w_1x_1^3 = -w_2x_2^3$ (8

We can divide (8) by (6) and the result is

$$x_1^2 = x_2^2 \quad \text{or} \quad x_1 = -x_2. \qquad (9$$

Use (9) and divide (6) by x_1 on the left and $-x_2$ on the right to get

$$w_1 = w_2. \qquad (10$$

Substituting (10) into (5) results in $w_1 + w_2 = 2$. Hence

$$w_1 = w_2 = 1. \qquad (11$$

Now using (11) and (9) in (7), we write

$$w_1x_1^2 + w_2x_2^2 = x_2^2 + x_2^2 = 2/3 \quad \text{or} \quad x_2^2 = 1/3 \qquad (12$$

Finally, from (12) and (9) we see that the nodes are

$$-x_1 = x_2 = 1/3^{1/2} \approx 0.5773502692.$$

We have found the nodes and weights that make up the two-point Gauss-Legendre rule. Since the formula is exact for cubic equations, the error term will involve the fourth derivative.

. • All polynomials of degree 3 or less will be ***exactly*** integrated with a Gauss-Legendre 2 point formula.

Example Gauss- Quad

- **Evaluate** $I = \int_0^4 xe^{2x}dx = 5216.926477$
- **Coordinate transformation**

$$x = \frac{b-a}{2}x_d + \frac{a+b}{2} = 2x_d + 2; \quad dx = 2dx_d$$

$$I = \int_0^4 xe^{2x}dx = \int_{-1}^1 (4x_d + 4)e^{4x_d+4}dx_d = \int_{-1}^1 g(x_d)dx_d$$

- **Two-point formula**

$$I = \int_{-1}^1 g(x_d)dx_d = g\left(\frac{-1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right) = \left(4 - \frac{4}{\sqrt{3}}\right)e^{4-\frac{4}{\sqrt{3}}} + \left(4 + \frac{4}{\sqrt{3}}\right)e^{4+\frac{4}{\sqrt{3}}}$$
$$= 9.167657324 + 3468.376279 = 3477.543936 \qquad (\varepsilon = 33.34\%)$$

- **Three-point formula**

$$I = \int_{-1}^1 g(x_d)dx_d = \frac{5}{9}g(-\sqrt{0.6}) + \frac{8}{9}g(0) + \frac{5}{9}g(\sqrt{0.6})$$
$$= \frac{5}{9}(4 - 4\sqrt{0.6})e^{4-\sqrt{0.6}} + \frac{8}{9}(4)e^4 + \frac{5}{9}(4 + 4\sqrt{0.6})e^{4+\sqrt{0.6}}$$
$$= \frac{5}{9}(2.221191545) + \frac{8}{9}(218.3926001) + \frac{5}{9}(8589.142689)$$
$$= 4967.106689 \qquad (\varepsilon = 4.79\%)$$

➢ Four-point formula

$$I=\int_{-1}^{1} g(x_d)\,dx_d = 0.34785\left[g(-0.861136)+g(0.861136)\right]$$
$$+0.652145\left[g(-0.339981)+g(0.339981)\right]$$
$$=5197.54375 \qquad (\varepsilon=0.37\%)$$

Table 1: Weighting factors *c* and function arguments *x* used in Gauss Quadrature Formulas.

Points	Weighting Factors	Function Arguments	Truncation Error
2	$c_0=1.0000000$	$x_0=-0.577350269$	$\cong f^{(4)}(\xi)$
	$c_1=1.0000000$	$x_1=0.577350269$	
3	$c_0=0.5555556$	$x_0=-0.774596669$	$\cong f^{(6)}(\xi)$
	$c_1=0.8888889$	$x_1=0.000000000$	
	$c_2=0.5555556$	$x_2=0.774596669$	
4	$c_0=0.3478548$	$x_0=-0.861136312$	$\cong f^{(8)}(\xi)$
	$c_1=0.6521452$	$x_1=-0.339981044$	
	$c_2=0.6521452$	$x_2=0.339981044$	
	$c_3=0.3478548$	$x_3=0.861136312$	
5	$c_0=0.2369269$	$x_0=-0.906179846$	$\cong f^{(10)}(\xi)$
	$c_1=0.4786287$	$x_1=-0.538469310$	
	$c_2=0.5688889$	$x_2=0.000000000$	
	$c_3=0.4786287$	$x_3=0.538469310$	
	$c_4=0.2369269$	$x_4=0.906179846$	
6	$c_0=0.1713245$	$x_0=-0.932469514$	$\cong f^{(12)}(\xi)$
	$c_1=0.3607616$	$x_1=-0.661209386$	
	$c_2=0.4679139$	$x_2=-0.238619186$	
	$c_3=0.4679139$	$x_3=0.238619186$	Gauss-Legendre
	$c_4=0.3607616$	$x_4=0.661209386$	Formulas
	$c_5=0.1713245$	$x_5=0.932469514$	

2nd Example

a) Use two-point Gauss Quadrature Rule to approximate the distance covered by a rocket from t=8 to t=30 as given by

$$x=\int_{8}^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

b) Find the true error, E_t for part (a).

c) Find the absolute relative true error for part (a).

Solution:

First, change the limits of integration from [8,30] to [-1,1]by previous relations as follows

$$\int_a^b f(x)dx=\int_{-1}^{1} f\left(\frac{b-a}{2}x_d+\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)dx_d=\int_{-1}^{1} g(x_d)dx_d$$

$$\int_{8}^{30} f(t)dt = \frac{30-8}{2}\int_{-1}^{1} f\left(\frac{30-8}{2}x + \frac{30+8}{2}\right)dx$$

$$= 11\int_{-1}^{1} f(11x+19)dx$$

Next, get weighting factors and function argument values from Table 1 for the two point rule

$C_1=W_1=1.00000$; $C_2=W_2=1.000000$; $X_1=-\,0.577350269$; $X_2=0.577350269$

Now we can use the Gauss Quadrature formula

Since

$$11\int_{-1}^{1} f(11x+19)dx \approx 11c_1 f(11x_1+19)+11c_2 f(11x_2+19)$$

$$=11f(11(-0.5773503)+19)+11f(11(0.5773503)+19)$$

$$=11f(12.64915)+11f(25.35085)$$

$$=11(296.8317)+11(708.4811)$$

$$=11058.44\ m$$

$$f(12.64915)=2000\,ln\left[\frac{140000}{140000-2100(12.64915)}\right]-9.8(12.64915)$$

$$=296.8317$$

$$f(25.35085)=2000\,ln\left[\frac{140000}{140000-2100(25.35085)}\right]-9.8(25.35085)$$

$$=708.4811$$

b) The true error, E_t , is

$$E_t = True\ Value - Approximate\ Value$$

$$=11061.34-11058.44$$

$$=2.9000\ m$$

c) The absolute relative true error, $|\epsilon_t| =$

$$\left|\frac{11061.34-11058.44}{11061.34}\right|\times 100\% \quad = 0.0262\%$$

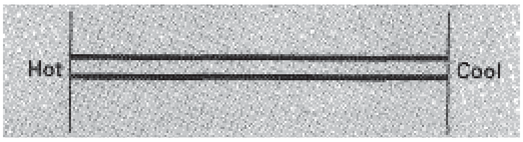
Lecture 6: Finite Difference: Parabolic Equations (chapra chp 30)

We now turn to the parabolic equations that are employed to characterize time-variable problems. In the latter part of this chapter, we will illustrate how this is done in two spatial dimensions for the heated plate. Before doing this, we will first show how the simpler one-dimensional case is approached.

1 . THE HEAT CONDUCTION EQUATION

FIGURE 30.1

A thin rod, insulated at all points except at its ends



At steady – state case, the present balance, considers the amount of heat stored in the element over a unit time period Δt; thus inputs – outputs = storage

$$q(x)\,\Delta y\,\Delta z\,\Delta t - q(x+\Delta x)\,\Delta y\,\Delta z\,\Delta t = \Delta x\,\Delta y\,\Delta z \rho C\,\Delta T$$

Dividing by the volume of the element $(=\Delta x\,\Delta y\,\Delta z)$ and Δt gives

$$\frac{q(x) - q(x+\Delta x)}{\Delta x} = \rho C \frac{\Delta T}{\Delta t}$$

Taking the limit yields

$$-\frac{\partial q}{\partial x} = \rho C \frac{\partial T}{\partial t}$$

Substituting Fourier's law of heat conduction [$q_i = -k\rho C \dfrac{\partial T}{\partial i}$); c=Sp. Heat cal/g.c^o,ρ=density g/cm^3, q_i=storage energy] results in

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \qquad (1)$$

where k is thermal conductivity of material.

2. EXPLICIT FD METHODS (AL- khafaji)

The steady state distribution for a given rod is given by the straight line connecting the boundary temp at the two ends (Fig 2).

The solution of eq. 30.1 is obtained by the first considering the initial and Boundary conditions:

i. at $x=x_0$ & $t_0 \le t \le \infty$; is $T(x_0, t) = T_0$ (2a)
ii. at $x=x_1$ & $t_0 \le t \le \infty$; is $T(x_1, t) = T_1$ (2b)
iii. at $t=t_0$ & $x_0 \le x \le x_1$; is $T(x, t_0) = T_x$ (2c)

Fig (2)

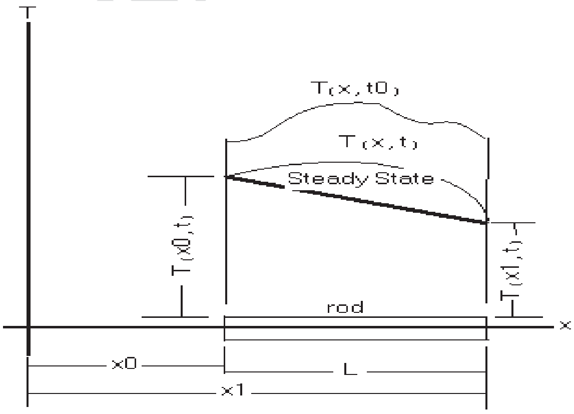
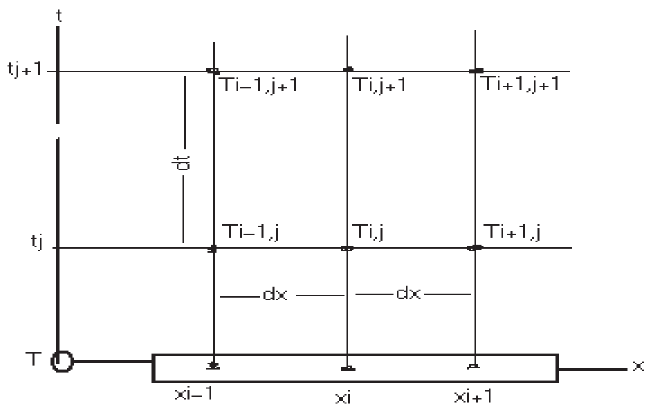


Fig. 3.



Considering Fig. 3, the solution of eq. 1. is attained by dividing the rod into equal segments and approximating difference as:

$$\frac{\partial^2 T}{\partial X^2} = \frac{1}{(\Delta x)^2}\left[T_{i-1,j} - 2T_{i,j} + T_{i+1,j}\right] \qquad (3)$$

$$\frac{\partial T}{\partial t} = \frac{1}{\Delta t}\left[-T_{i,j} + T_{i,j+1}\right] \qquad (4)$$

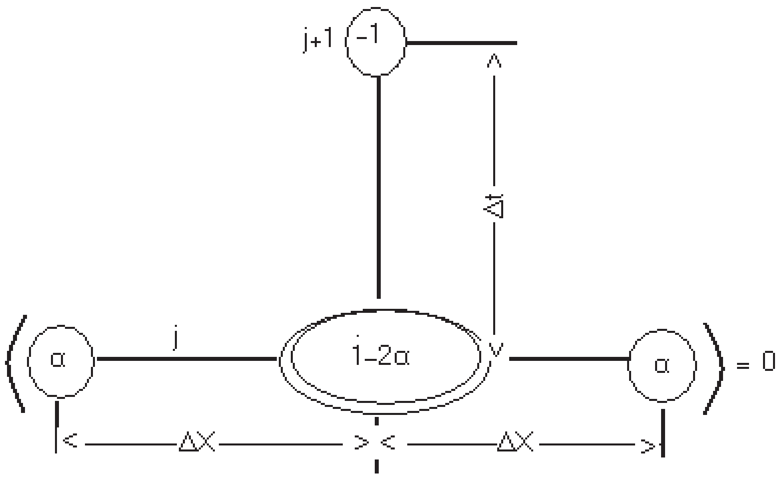
Hence eq.1. becomes

$$T_{i,j+1} = \alpha T_{i-1,j} + (1-2\alpha)T_{i,j} + \alpha T_{i+1,j} \qquad (5a)$$

Where $\alpha = \dfrac{k\Delta t}{(\Delta x)^2}$ (5b)

for stability $0 \le \alpha \le \frac{1}{2}$ (5 c)

Eq. 5a. can be expressed in Stencil Form as: (which is useful for hand calculation.)



(6)

Example 1: Thin rod of length 10 cm has the following values: at t=0 the temp. of it is zero & B.C.of $T(0)=100^o_C$, $T(10)= 50^o_C$ are fixed for all time. Use ΔX= 2 cm, Δt=0.1 sec. and k=0.835 cm^2/sec. to determine the temperature distribution for $0 \le t = 0.3\ sec.$

Solution:

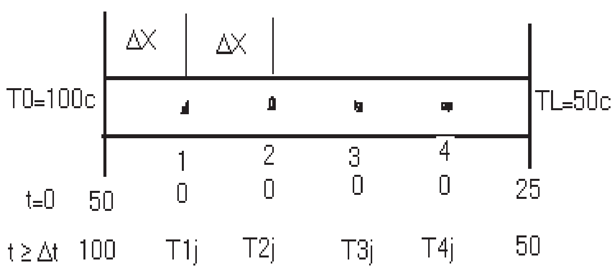
We have to begine with investigation that $\alpha = \dfrac{k\Delta t}{(\Delta x)^2}$ is ensure a stable solution. $\alpha = \frac{0.835(0.1)}{(2)^2} =$

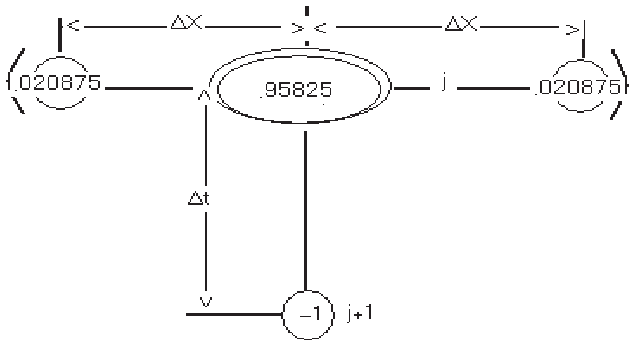
0.020875 ≤ 1/2 (ok)

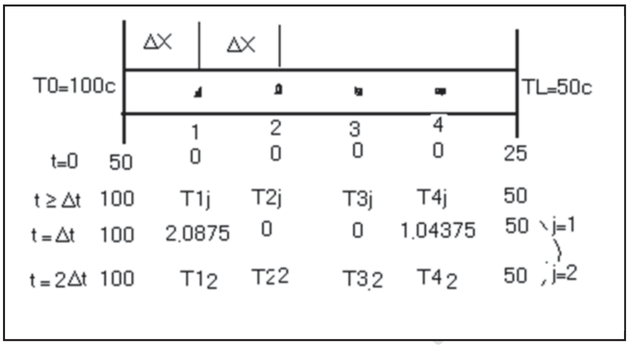
Now adjusting the B. C. for all increments;

For t=Δt use ; at $x=x_0 \rightarrow T_0 = \frac{100+0}{2} = 50$; at $x = x_L \rightarrow T_L = \frac{50+0}{2} = 25$

For t > Δt use at $x=x_0 \rightarrow T_0=100^o c$; at $x = x_L \rightarrow T_L = 50^o c$







at $t=\Delta t$ j=1

T_{11}= .020875(100) + .95825(0) + .020875(0) = 2.0875

T_{21} =.020875(0) + .95825(0) + .020875(0) = 0.0

T_{31} =.020875(0) + .95825(0) + .020875(0) = 0.0

T_{41} =.020875(0) + .95825(0) + .020875(50) = 1.04375

at $t=2\Delta t$ j=2

T_{12}= .020875(100) + .95825(2.0875) + .020875(0) = 4.08785

T_{22} =.020875(2.0875) + .95825(0) + .020875(0) = 0.043577

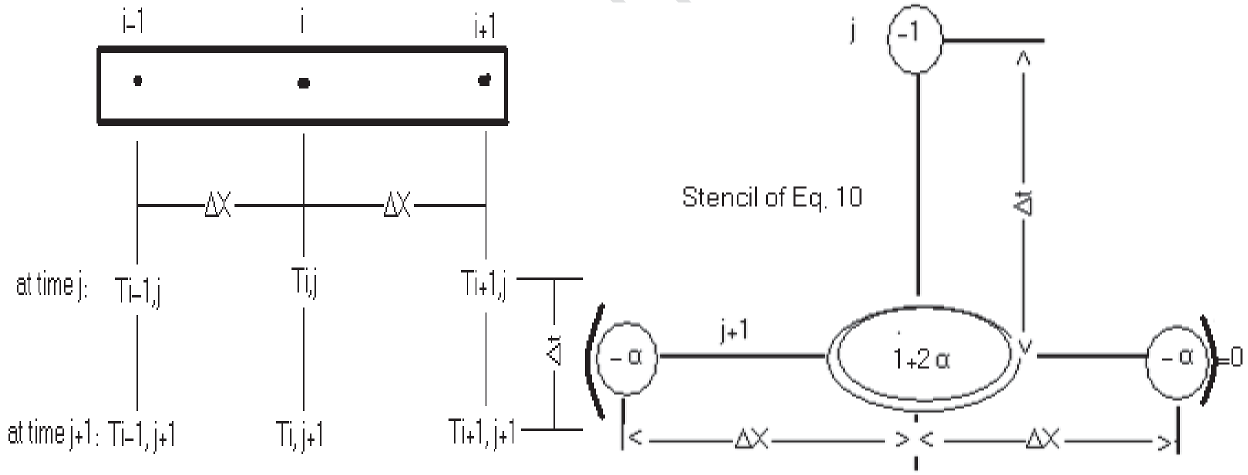
T_{32} =.020875(0) + .95825(0) + .020875(1.04375) = 0.021788

T_{42} =.020875(0) + .95825(1.04375) + .020875(50) = 2.044

and so on at $t= 3\Delta t$ use j=3 to find T_{13}= 6.0056; T_{23}= 0.12755; T_{33}= 0.06446; T_{43}= 3.00287

3. Implicit Method

It does not have any of the disadvantages of Explicit method for $\alpha > \frac{1}{2}$ and a very small time increment, but does involve solving a set of linear Algebraic equations. The development of the difference equation is realized by noting that the nodal value at time **t** is directly influenced by that at time **t+Δt**. This is explained by considering the following Rod segment.



For time j & j+1, the partial derivative $\frac{\partial^2 T}{\partial X^2}$ is taken as:

$$\frac{\partial^2 T}{\partial x^2}=\frac{1}{(\Delta x)^2}\left[T_{i-1,j}-2T_{i,j}+T_{i+1,j}\right](1-\theta)+\frac{1}{(\Delta x)^2}\left[T_{i-1,j+1}-2T_{i,j+1}+T_{i+1,j+1}\right]\theta \qquad (7)$$

where θ is a weighting factor , $0 \le \theta \le 1$.

Hence $$\frac{\partial T}{\partial t}=\frac{1}{\Delta t}\left[-T_{i,j}+T_{i,j+1}\right] \qquad (8)$$

Using eqs. (7 & 8) to find Heat conduction eq. as

$$\frac{c}{(\Delta x)^2}\left[T_{i-1,j}-2T_{i,j}+T_{i+1,j}\right](1-\theta)+\frac{c}{(\Delta x)^2}\left[T_{i-1,j+1}-2T_{i,j+1}+T_{i+1,j+1}\right]\theta=\frac{1}{\Delta t}\left[-T_{i,j}+T_{i,j+1}\right];$$

after simplifying;

$$\alpha(1-\theta)T_{i-1,j}+(1\text{-}2\alpha+2\alpha\theta)T_{i,j}+\alpha(1-\theta)T_{i+1,j}+\alpha\theta T_{i-1,j+1}-(2\alpha\theta+1)\,T_{i,j+1}+\alpha\theta T_{i+1,j+1}=0 \quad (9)$$

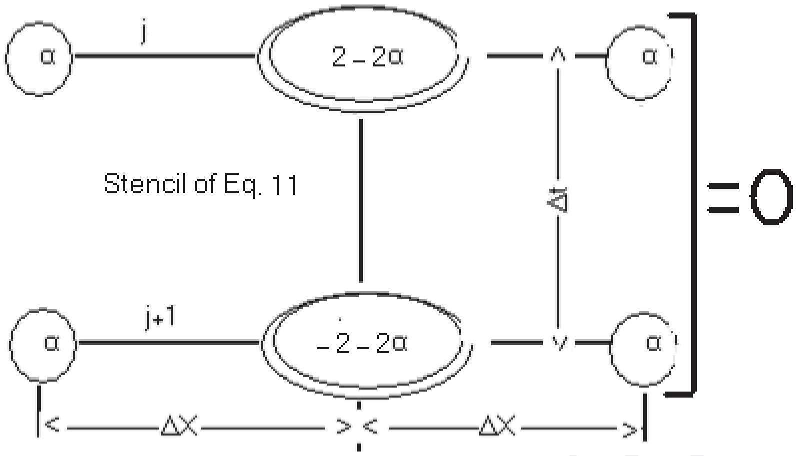
It's greater accuracy is attained when $\frac{1}{2} < \alpha < 1$. Eq. 9 is implicit because it involves more than one unknown at any step (j+1) in the time domain. It also , can be defined as general formula because θ.

- When $\theta = 0$; eq.9 becomes identical to the classic Explicit Formula 5a.
- When $\theta = 1$; it becomes $T_{i,j} = -\alpha T_{i-1,j+1} + (2\alpha + 1)\, T_{i,j+1} - \alpha T_{i+1,j+1}$ (10); ***Which is called the backward implicit approximation & it can be seen as stencil form in above fig.***

4 . Crank – Nicolson implicit method

Return back to eq. 9, when it uses $\theta = 0.5$ it becomes the well Known Crank – Nicolson formula as:

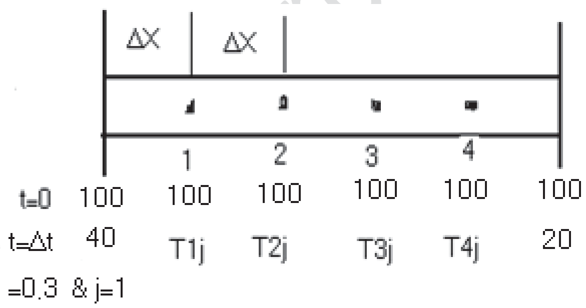
$\alpha T_{i-1,j} + (2 - 2\alpha) T_{i,j} + \alpha T_{i+1,j} + \alpha T_{i-1,j+1} - (2\alpha + 2)\, T_{i,j+1} + \alpha T_{i+1,j+1} = 0$ (11); ***which can be seen as stencil form in following fig.***

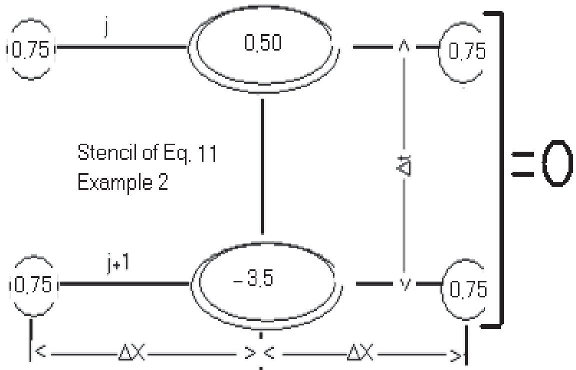


Example 2: Determine the temperature through out a rod 10 ft. long with B. Temp of 20 & 40 °F. The initial temp of the rod is 100°F with K= 10 ft²/ min. Let ΔX= 2 ft, use Crank – Nicolson implicit scheme for Δt=0.3 min.

Solution: $\alpha = \frac{10(0.3)}{(2)^2} = 0.75 < 1$ { *OK for implicit $0.5 < \alpha < 1$* }

Initial B. Conditions.





Applying for the four nodes at t=0.3 gives

Node 1: (0.75 + 0.50 + 0.75) (100) + (0.75) 40 – 3.5 T_{11} + (0.75) T_{21} = 0

Node 2: (0.75 + 0.50 + 0.75) (100) + (0.75) T_{11} – 3.5 T_{21} + (0.75) T_{31} = 0

Node 3: (0.75 + 0.50 + 0.75) (100) + (0.75) T_{21} – 3.5 T_{31} + (0.75) T_{41} = 0

Node 4: (0.75 + 0.50 + 0.75) (100) + (0.75) T_{31} – 3.5 T_{41} + (0.75) 20 = 0 ; In Matrix Form

$$\begin{bmatrix} -3.5 & 0.75 & 0 & 0 \\ 0.75 & -3.5 & 0.75 & 0 \\ 0 & 0.75 & -3.5 & 0.75 \\ 0 & 0 & 0.75 & -3.5 \end{bmatrix} \begin{Bmatrix} T_{11} \\ T_{21} \\ T_{31} \\ T_{41} \end{Bmatrix} = - \begin{Bmatrix} 200+30 \\ 200 \\ 200 \\ 200+15 \end{Bmatrix}$$

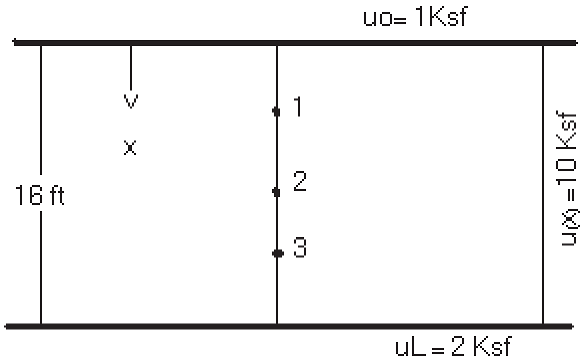
Solving for the Unknown T_{i1} gives; $T_{11} = 86.296\ ^oF$; $T_{21} = 96.048^oF$; $T_{31} = 95.262^oF$; $T_{41} = 81.842^oF$.

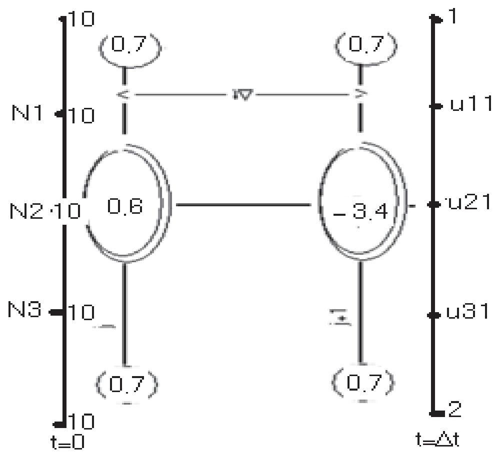
Example 3: (for civil Eng.)

Determine the excess pore water pressure distribution for the following soil strata after two days, using the implicit scheme with $\Delta x = 4$ ft & $\Delta t = 1$ day & $C = 11.2\ ft^2/day$.

Solution; Replacing temp. T with pore water pressure u ; and calculate $\alpha = \frac{C\Delta t}{(\Delta t)^2} = \frac{11.2\,(1)}{(4)^2} = 0.7$ { OK}

The Stencil for the 1st time increment is followed.





N1: $10(0.7) + 10\,(0.6) + 10\,(0.7) + 1\ \ (0.7) - 3.4\,u_{11} + 0.7\,u_{21} = 0$

N2: $10(0.7) + 10\,(0.6) + 10\,(0.7) + u_{11}(0.7) - 3.4\,u_{21} + 0.7\,u_{31} = 0$

N3: $10(0.7) + 10\,(0.6) + 10\,(0.7) + u_{21}(0.7) - 3.4\,u_{31} + 0.7\,(2) = 0$

Expressing these Eqs. in Matrix form;

$$\begin{bmatrix} -3.4 & 0.7 & 0 \\ 0.7 & -3.4 & 0.7 \\ 0 & 0.7 & -3.4 \end{bmatrix} \begin{Bmatrix} U_{11} \\ U_{21} \\ U_{31} \end{Bmatrix} = - \begin{Bmatrix} 20.7 \\ 20.0 \\ 21.4 \end{Bmatrix}$$

Solving yields $U_{11}= 7.985$; $U_{21}= 9.2123$; $U_{31}= 8.191$. Use these results for the second time ($2\Delta t$) increment to establishing the following eqs.

N1: $1\,(0.7) \quad + 7.985\,(0.6) + 9.213\,(0.7) + 1\ \ (0.7) - 3.4\,u_{12} + 0.7\,u_{22} = 0 \ \rightarrow 12.64 - 3.4\,u_{12} + 0.7\,u_{22} = 0$

N2: $7.985(0.7) + 9.213\,(0.6) + 8.191\,(0.7) + u_{12}(0.7) - 3.4\,u_{22} + 0.7\,u_{32} = 0 \rightarrow 16.851 + .7u_{12} - 3.4\,u_{22} + .7u_{32} = 0$

N3: $9.213(0.7) + 8.191\,(0.6) + 2\ \ (0.7) \quad + u_{22}(0.7) - 3.4\,u_{32} + 0.7\,(2) = 0 \ \rightarrow 14.164 + 0.7\,u_{22} - 3.4u_{32} = 0$

In Matrices form

$$\begin{bmatrix} -3.4 & 0.7 & 0 \\ 0.7 & -3.4 & 0.7 \\ 0 & 0.7 & -3.4 \end{bmatrix} \begin{Bmatrix} U_{12} \\ U_{22} \\ U_{32} \end{Bmatrix} = - \begin{Bmatrix} 12.64 \\ 16.851 \\ 14.164 \end{Bmatrix}$$

Solving to find $U_{12}= 5.198$; $U_{22}= 7.189$; & $U_{32}= 5.646$

5. PE in Two Spatial Dimensions:

For two dimensions, the Heat- Conduction equation is

$$\frac{\partial T}{\partial t} = K\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] \qquad (12)$$

One application of this equation is to model the temperature distribution on the face of a heated plate. By substituting Finite- Difference approximation of the form of eqs. 3&4 in eq. 12 to find the Explicit Form:

$T_i^{l+1} = T_i^l + 2\,\alpha\,[\,T^l_{i+1} - 2\,T_i^l + T^l_{i-1}]$ (13) ≡ eq. (5)

for uniform grid ($\Delta x = \Delta y$). While Implicit Forms, eqs. (10 & 11) can be rearranged as:

$-2\,\alpha\, T_{i-1}^{l+1} + T_i^{l+1}(1 + 4\alpha) - 2\,\alpha\, T_{i+1}^{l+1} = T_i^l$ (14) ≡ eq. (10)

$(1 + 4\alpha)T_1^{l+1} - 2\,\alpha T_2^{l+1} = T_1^l + 2\alpha\, f_{0(t^{l+1})}$ (15) is for left end of the rod (i=0), while

for the last node (i=m); $-2\,\alpha\, T_{m-1}^{l+1} + (1 + 4\alpha)T_m^{l+1} = T_m^l + 2\alpha f_{m+1}(t^{l+1})$ (16)

H. Works: *Use chapra notations to Derive all eqs. (13, 14, 15 & 16)*

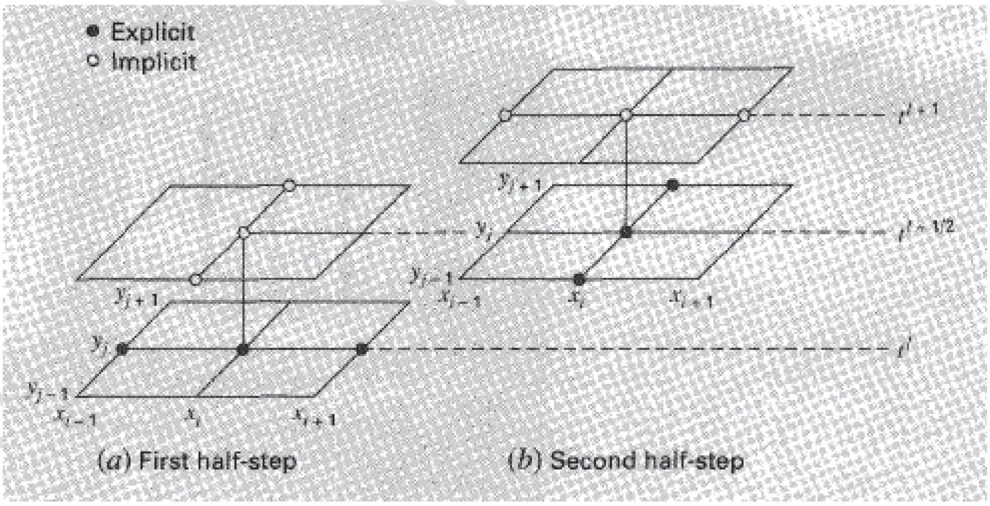
6. The Alternating – Direction Implicit (ADI) Scheme: *(Chapra chp 30)*

*The alternating-direction implicit, or ADI, scheme provides a means for solving parabolic equations in two spatial dimensions using tridiagonal matrices. To do this, each time in*crement is executed in two steps (Fig. 6). For the first step, Eq. (12) is approximated by

$$\frac{T_{i,j}^{l+1/2} - T_{i,j}^{l}}{\Delta t/2} = k\left[\frac{T_{i+1,j}^{l} - 2T_{i,j}^{l} + T_{i-1,j}^{l}}{(\Delta x)^2} + \frac{T_{i,j+1}^{l+1/2} - 2T_{i,j}^{l+1/2} + T_{i,j-1}^{l+1/2}}{(\Delta y)^2}\right] \qquad (17)$$

---------------- Explicitly ----------------------------------

Fig. (6) The two half-steps used In irnplementing the alternating-direction implicit scheme for solving parabolic equations in two spatial dimensions.



For case of square grid ($\Delta x = \Delta y$), eq. 17 can be expressed as;

$$-\alpha T_{i,j-1}^{l+\frac{1}{2}} + 2(1+\alpha)T_{i,j}^{l+\frac{1}{2}} - \alpha T_{i,j+1}^{l+\frac{1}{2}} = \alpha T_{i-1,j}^{l} + 2\,(1-\alpha\,)T_{i,j}^{l} + \alpha\, T_{i+1,j}^{l} \qquad (18)$$

which, when written for the system, results in Tridiagonal set of simultaneous equations.

For 2nd step eq. 12 is approximated by

$$\frac{T_{i,j}^{l+1} - T_{i,j}^{l+1/2}}{\Delta t/2} = k\left[\frac{T_{i+1,j}^{l+1} - 2T_{i,j}^{l+1} + T_{i-1,j}^{l+1}}{(\Delta x)^2} + \frac{T_{i,j+1}^{l+1/2} - 2T_{i,j}^{l+1/2} + T_{i,j-1}^{l+1/2}}{(\Delta y)^2}\right] \qquad (19)$$

--------------------Implicitly --------------------------------

For case of square grid ($\Delta x = \Delta y$), eq. 19 can be expressed as;

$$\alpha T_{i,j-1}^{l+\frac{1}{2}} + 2(1-\alpha)T_{i,j}^{l+\frac{1}{2}} + \alpha T_{i,j+1}^{l+\frac{1}{2}} = -\alpha T_{i-1,j}^{l+1} + 2\,(1+\alpha)T_{i,j}^{l+1} - \alpha\, T_{i+1,j}^{l+1} \qquad (20)$$

which, also when written for 2D grid, it's results in a Tridiagonal system as Fig. 7.

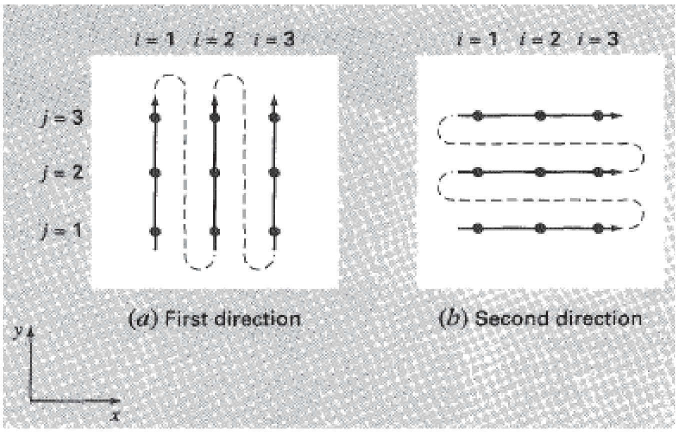


Fig (7) . The AD1 method only results in tridiagonal equations, if it is applied along the dimension that is implicit. Thus, on the first step (a), it is applied along the y dimension and, on the second step (b), along the *x* dimension. These "alternating directions" are the root of the method's name.

Eq. 18 can be rearranged to be used for the 1st direction, after dividing by α it becomes;

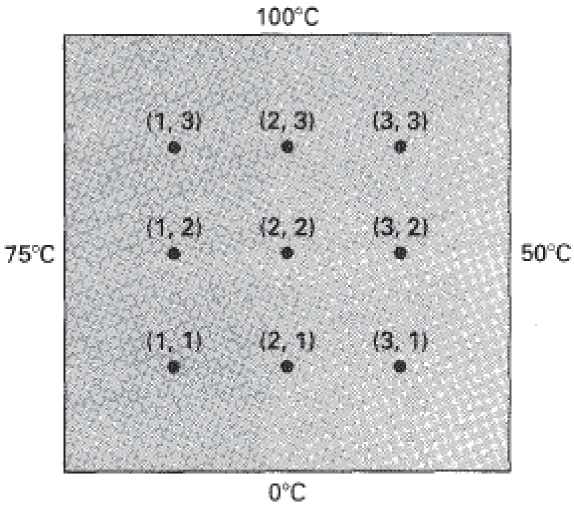
$$-T_{i,j-1}^{l+\frac{1}{2}} + \left(\frac{2}{\alpha}+2\right)T_{i,j}^{l+\frac{1}{2}} - T_{i,j+1}^{l+\frac{1}{2}} = T_{i-1,j}^{l} + \left(\frac{2}{\alpha}-2\right)T_{i,j}^{l} + T_{i+1,j}^{l} \qquad (21)$$

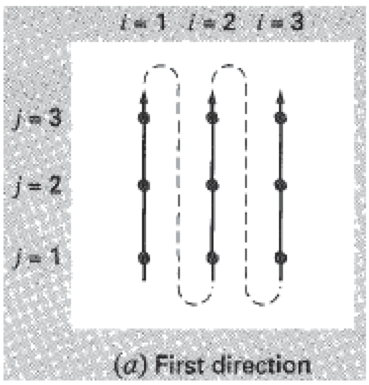
While for 2nd direction Eq. 20 becomes;

$$-T_{i-1,j}^{l+1} + \left(\frac{2}{\alpha}+2\right)T_{i,j}^{l+1} - T_{i+1,j}^{l+1} = T_{i,j-1}^{l+\frac{1}{2}} + \left(\frac{2}{\alpha}-2\right)T_{i,j}^{l+\frac{1}{2}} + T_{i,j+1}^{l+\frac{1}{2}} \qquad (22)$$

Example (ADI):

Plate of Aluminum (40 * 40) cm, Determine heat flux for the heated plate in the following Fig. Assuming at t=0 the temp is zero, the B.C. at (t ≥ Δt) are instantaneously brought to the levels shown in the Fig. Employ Δt = 10 sec, k= 0.835 cm^2/sec.





Solution: $\alpha = \frac{k\Delta t}{(\Delta t)^2} = \frac{.835\,(10)}{(10)^2} = 0.0835$; Eq. 21 has to be applied in 1st step to find temp at (1,1; 1,2; & 1,3) in the figs. above.

$-T_{i,j-1}^{1} + \left(\frac{2}{\alpha}+2\right)T_{i,j}^{1} - T_{i,j+1}^{1} = T_{i-1,j}^{0} + \left(\frac{2}{\alpha}-2\right)T_{i,j}^{0} + T_{i+1,j}^{0}$ putting $\frac{2}{\alpha} + 2 = 25.95$ & $\frac{2}{\alpha} - 2 = 21.95$

<u>at t=5 sec 1st step</u> for T_{11}

$-T^1_{1,0} + 25.95\, T^1_{1,1}$ - $T^1_{1,2}$ $= T^0_{0,1} +\ 21.95\, T^0_{1,1} +\ T^0_{2,1} \rightarrow$ 0 + 25.95 $T_{11} - T_{12}$ = 75 + 21.95 (0) + 0

For T_{12}

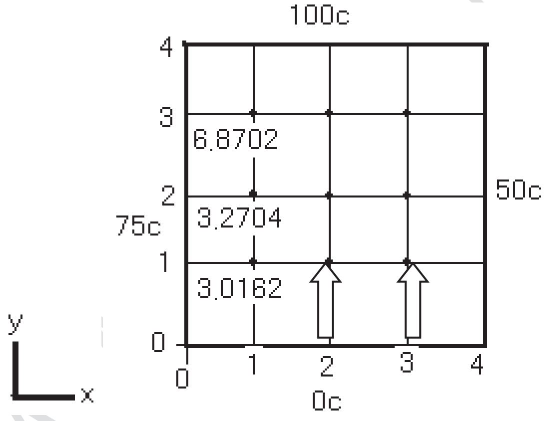
$-T^1_{1,1} + 25.95\, T^1_{1,2}$ - $T^1_{1,3}$ $= T^0_{0,2} +\ 21.95\, T^0_{1,2} +\ T^0_{2,2} \rightarrow$ - T_{11} + 25.95 $T_{12} - T_{13}$ = 75 + 21.95 (0) + 0

For T_{13}

$-T^1_{1,2} + 25.95\, T^1_{1,3}$ - $T^1_{1,4}$ $= T^0_{0,3} +\ 21.95\, T^0_{1,3} +\ T^0_{2,3} \rightarrow$ - T_{12} + 25.95 T_{13} – 100 = 75 + 21.95 (0) + 0

in matrix form $\begin{bmatrix} 25.95 & -1 & 0 \\ -1 & 25.95 & -1 \\ 0 & -1 & 25.95 \end{bmatrix} \begin{Bmatrix} T_{11} \\ T_{12} \\ T_{13} \end{Bmatrix} = \begin{Bmatrix} 75 \\ 75 \\ 175 \end{Bmatrix}$ solving by any way to find T_{11}= 3.0162;

T_{12}=3.2704; T_{13}=6.8702. In a similar fashion, tridiagonal equations can be developed



$$\begin{bmatrix} 25.95 & -1 & 0 \\ -1 & 25.95 & -1 \\ 0 & -1 & 25.95 \end{bmatrix} \begin{Bmatrix} T_{21} \\ T_{22} \\ T_{23} \end{Bmatrix} = \begin{Bmatrix} 3.0162 \\ 3.2704 \\ 106.87 \end{Bmatrix}$$

Solving to find T_{21}= 0.1274; T_{22}=0.2900; T_{23}= 4.1294; and T_{31}=2.0181; T_{32}= 2.2423; T_{33}=6.026

Step 2: *(t= 2*5= 10 sec) for 2nd direction eq. 22 is applied to nodes { 1,1; 2,1; & 3,1 }*

$-T^2_{i-1,j} + \left(\frac{2}{\alpha} + 2\right) T^2_{i,j} -\ \ T^2_{i+1,j}\ =\ T^1_{i,j-1} + (\ \frac{2}{\alpha}$ - 2 $)T^1_{i,j}\ + T^1_{i,j+1}$

For T_{11}

$-T^2_{0,1} + 25.95\ \ T^2_{1,1} -\ \ T^2_{2,1}\ =\ T^1_{1,0} + 21.95\ \ T^1_{1,1}\ + T^1_{1,2}$

- 75 +25.95 $T_{11} - T_{21}$ = 0 + 21.95 (3.0162) + 3.27

25.95 $T_{11} - T_{21}$ = 144.48 …………(i)

For T_{21}

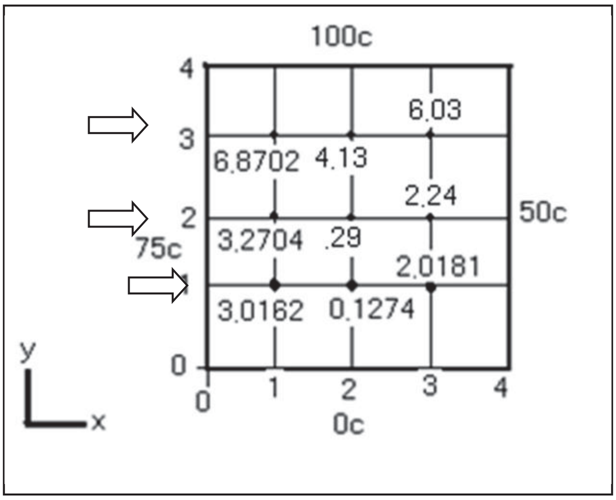
- T_{11} +25.95 $T_{21} - T_{31}$ = 0 + 21.95 (.127) + 0.29

- T_{11} +25.95 $T_{21} - T_{31}$ = 3.09 …… (ii)

For T_{31}

- T_{21} +25.95 $T_{31} - T_{41}$ = 0 + 21.95 (2.018) + 2.24

- T_{21} +25.95 T_{31} = 46.58+50 ………(iii)



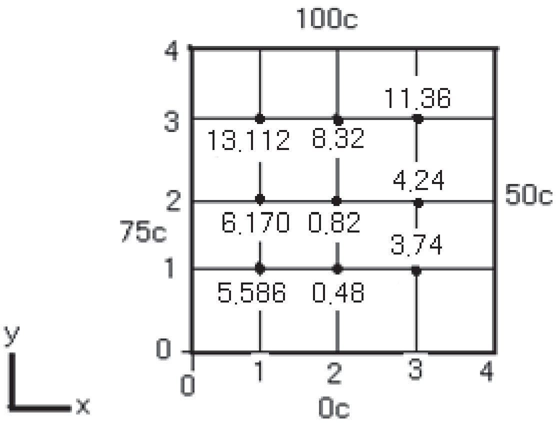
For such tridiagonal equations, the following Matrix Form can be developed;

$\begin{bmatrix} 25.95 & -1 & 0 \\ -1 & 25.95 & -1 \\ 0 & -1 & 25.95 \end{bmatrix} \begin{Bmatrix} T_{11} \\ T_{21} \\ T_{31} \end{Bmatrix} = \begin{Bmatrix} 144.48 \\ 3.09 \\ 96.58 \end{Bmatrix}$ solving to find T_{11}= 5.586; T_{21}= 0.479; T_{31}= 3.74

The computation have to be completed for the remained Nodes, which yields to:

T_{12} = 6.168 , T_{22} = 0.824 , T_{32} = 4.24 ; T_{13} = 13.112 , T_{23} = 8.321, T_{33} = 11.361;

at this stage the heat flux can be shown in the following fig.



Then reworking 1st direction and then 2nd direction till the wanted time.