Lecture 7: FEM & RWmethod (Chapra 31)

1. THE GENERAL APPROACH

Although the particulars will vary, the implementation of the finite-element approach usually follows a standard step-by-step procedure. The following provides a brief overview of each of these steps. The application of these steps to engineering problem contexts will be developed in subsequent sections.

1.1.Discretization:

This step involves dividing the solution domain into finite elements. Figure (1) provides examples of elements employed in one, two, and three dimensions. The points of intersection of the lines that make up the sides of the elements are referred to as nodes and the sides themselves are called nodal lines or planes.

Nodes: are the points of intersection of the lines that make up the sides of the elements.

Nodal lines: are referred to the sides of the elements, into two dimensions.

Nodal plane: are referred to the sides of the elements, into three dimensions.

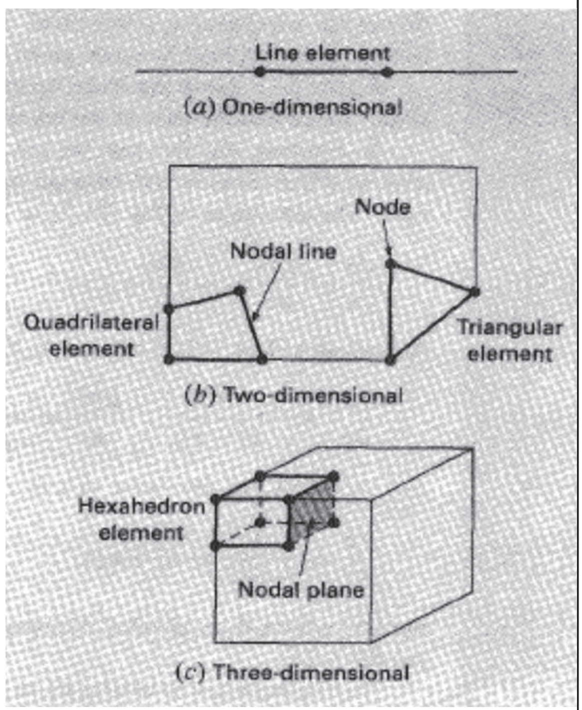


Figure (1)
Examples of elements employed in (a) one, (b) two, and (c) three dimensions

1.2. Element Equations

The next step is to develop equations to approximate the solution for each element. This involves two steps.

First, we must choose an appropriate function with unknown coefficients that will be used to approximate the solution.

Second, we evaluate the coefficients so that the function approximates the solution in an optimal fashion.

Choice of Approximation Functions. Because they are easy to manipulate mathematically, polynomials are often employed for this purpose. **For the one-dimensional case**, the simplest alternative is a first-order polynomial or straight line,

$$u(x) = a_0 + a_1 x \qquad (1)$$

This fun. must pass through the values of $u(x)$ at the end points of elements at x_1 & x_2, therefore;

$$u_1 = a_0 + a_1 x_1 = u(x_1)$$
$$u_2 = a_0 + a_1 x_2 = u(x_2)$$

These equations can be solved using Cramer's rule for

$$a_0 = \frac{u_1 x_2 - u_2 x_1}{x_2 - x_1} \qquad a_1 = \frac{u_2 - u_1}{x_2 - x_1}$$

These results can then be substituted into Eq. (1) which, after collection of terms, can be written as

$$U = N_1 u_1 + N_2 u_2 \qquad (2) \qquad \{\text{Approximation or Shape Fun.}\}$$

Where $N_1 = \frac{x_2 - x}{x_2 - x_1}$ (3) and $N_2 = \frac{x - x_1}{x_2 - x_1}$ (4) { interpolation Funs.}

eq. (2) in fact, is the Lagrange 1st order interpolating polynomial.

The derivative of Eq. (2) is

$$\frac{du}{dx} = \frac{dN_1}{dx}u_1 + \frac{dN_2}{dx}u_2 \qquad (5)$$

According to Eqs. (3) and (4), the derivatives of the N's can be calculated as

$$\frac{dN_1}{dx} = -\frac{1}{x_2 - x_1} \qquad \frac{dN_2}{dx} = \frac{1}{x_2 - x_1} \qquad (6)$$

therefore, the derivative of u is

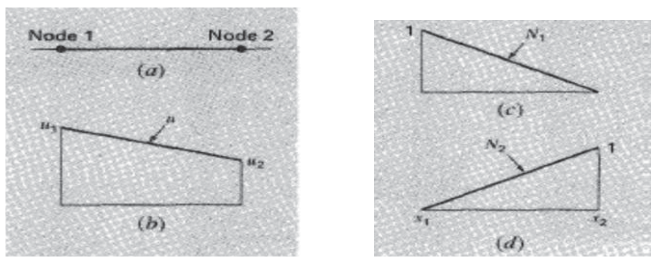


Figure (2)
(b) A linear approximation, or shape function for (a) a line element. The corresponding interpolation functions are shown in (c) and (d).

$$\frac{du}{dx} = \frac{1}{x_2 - x_1}(-u_1 + u_2) \qquad (7)$$

The integral can be expressed as

$$\int_{x_1}^{x_2} u\,dx = \int_{x_1}^{x_2} N_1 u_1 + N_2 u_2\,dx$$

Each term on the right-hand side is merely the integral of a right triangle with base $(x_2 - x_1)$ and height u. That is,

$$\int_{x_1}^{x_2} Nu\,dx = \frac{1}{2}(x_2 - x_1)u$$

Thus, the entire integral is

$$\int_{x_1}^{x_2} u\,dx = \frac{u_1 + u_2}{2}(x_2 - x_1) \qquad (8)$$

In other words, it is simply the trapezoidal rule.
Mathematically, the resulting element equations will often consist of a set of linear algebraic equations that can be expressed in matrix form,

$$[k]\{u\} = \{F\} \qquad (9)$$

where [k] = an *element property* or *stiffness matrix*, *{ u }* = a column vector of unknowns at the nodes, and {F} = a column vector reflecting the effect of any external influences applied at the nodes.

1.3. Assembly

The assembly process is governed by the concept of continuity. That is, the solutions for contiguous elements are matched so that the unknown values (and sometimes the derivatives) at their common nodes are equivalent. Thus, the total solution will be continuous.
When all the individual versions of Eq. (9) are finally assembled, the entire system is expressed in matrix form as

$$[k]\{\acute{u}\} = \{\acute{F}\} \qquad (10)$$

where [K] = the *assemblage property matrix* and $\{\acute{u}\}$ and $\{\acute{F}\}$ = column vectors for unknowns and external forces that are marked with primes to denote that they are an assemblage of the vectors $\{u\}$ and $\{F\}$ from the individual elements.

1.4. Boundary Conditions

Before Eq. (10) can be solved, it must be modified to account for the system's boundary conditions. These adjustments result in

$$[\bar{k}]\{u'\} = \{\bar{F}'\} \qquad (11)$$

where the over bars signify that the boundary conditions have been incorporated.

1.5. Solution

Solutions of Eq. (11) can be obtained with techniques described previously, such as *LU* decomposition. In many cases, the elements can be configured so that the resulting equations are banded. Thus, the highly efficient solution schemes available for such systems can be employed.

1.6. Postprocessing

In the following section, we illustrate how they can be applied to obtain numerical results for a simple physical system-***a heated rod***.

2. FINITE-ELEMENT APPLICATION IN *ONE DIMENSION*

Figure *(3)* shows a system that can be modeled by a one-dimensional form of Poisson's equation

$$\frac{d^2T}{dx^2} = -f(x) \qquad (12)$$

where *f(x)* = a function defining a heat source along the rod and where the ends of the rod are held at fixed temperatures,

$T(0,t)=T_1$ & $T(L,t)=T_2$

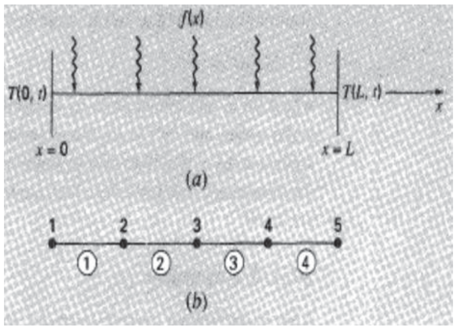


Figure (3)
(a) A long, thin rod subject to fixed boundary conditions and a continuous heat source along its axis. (b) The finite-element representation consisting of four equal-length elements and five nodes.

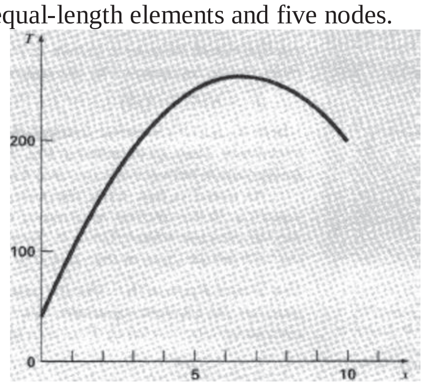


Figure (4)
The temperature distribution along a heated rod subject to a uniform heat source and held at fixed end temperatures.

EXAMPLE (1): Analytical Solution for a Heated Rod

Solve Eq. (12) for a 10-cm rod with boundary conditions of $T(0, t) = 40$ and $T(10, t) = 200$ and a uniform heat source of $f(x) = 10$.

Solution. The equation to be solved is

$$\frac{d^2T}{dx^2} = -10$$

Assume a solution of the form

$T = ax^2 + bx + c$

which can be differentiated twice to give T'' = $2a$. Substituting this result into the differential equation gives $a = -5$. The boundary conditions can be used to evaluate the remaining coefficients. For the first condition at $x = 0$, $40 = -5(0)^2 + b(0) + c$ or $c = 40$.

Similarly, for the second condition, $200 = -5(10)^2 + b(10) + 40$ which can be solved for $b = 66$. Therefore, the final solution is $T = -5x^2 + 66x + 40$. The results are plotted in Fig. (4).

2.1. Discretization

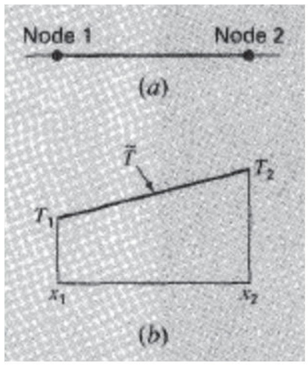
A simple configuration to model the system is a series of equal-length elements (Fig. 3 *b*). Thus, the system is treated as four equal-length elements and five nodes.

2.2. Element Equations

An individual element is shown in Fig. (5- a). The distribution of temperature for the element can be represented by the approximation function

$$\tilde{T} = N_1T_1 + N_2T_2 \qquad (13)$$

where N_1 and $\boldsymbol{N_2}$ = linear interpolation functions specified by Eqs. (3) and (4), respectively. Thus, as depicted in Fig. (5-b), the approximation function amounts to a linear interpolation between the two nodal temperatures.



Figure(5)

*(a)*A n individual element.

(b) The approximation function used to characterize the temperature distribution a long the element.

There are a variety of approaches for developing the element equation. In this section, we employ two of these. **First**, **a *direct approach*** will be used for the simple case where f (x) = 0.

$$q = -k'\frac{dT}{dx} \qquad \{\text{ Fourier's law }\}$$

where q = flux [cal/(cm^2.s)] and $\boldsymbol{K}$ = the coefficient of thermal conductivity [cal/(s.cm.^{o}C)]. If a linear approximation function is used to characterize the element's temperature, the heat flow into the element through node 1 can be represented by

$$q_1 = k'\frac{T_1 - T_2}{x_2 - x_1}$$

where q_l is heat flux at node 1. Similarly, for node 2,

$$q_2 = k'\frac{T_2 - T_1}{x_2 - x_1}$$

They can be simplified further by recognizing that Fourier's law can be used to couch the end fluxes themselves in terms of the temperature gradients at the boundaries. That is,

$$q_1 = -k'\frac{dT(x_1)}{dx} \qquad q_2 = k'\frac{dT(x_2)}{dx}$$

which can be substituted into the element equations to give

$$\frac{1}{x_2 - x_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{dT(x_1)}{dx} \\ \dfrac{dT(x_2)}{dx} \end{Bmatrix} \qquad (14)$$

Notice that Eq. (14) has been cast in the format of Eq. (9). Thus, we have succeeded in generating a matrix equation that describes the behavior of a typical element in our system. Because of general applicability of **a *direct approach*** in engineering, we will devote most of the section to the *method of weighted residuals.*

Second The Method of Weighted Residuals. The differential equation (12) can be re-expressed as

$$q = \frac{d^2T}{dx^2} + f(x)$$

The approximate solution [Eq. (13)] can be substituted into this equation. Because Eq. (13) is not the exact solution, the left side of the resulting equation will not be zero but will equal a residual,

$$R = \frac{d^2\tilde{T}}{dx^2} + f(x) \qquad (15)$$

The method of weighted residuals (MWR) consists of finding a minimum for the residual according to the general formula

$$\int_D RW_i \, dD = 0 \qquad i = 1, 2, \ldots, m \qquad (16)$$

where D = the solution domain and the W_i = linearly independent weighting functions.
At this point, there are a variety of choices that could be made for the weighting function (**see 3. MWR**). The most common approach for the finite-element method is to employ the interpolation functions N_i as the weighting functions. When these are substituted into Eq. (16), the result is referred to as Galerkin's method,

$$\int_D RN_i \, dD = 0 \qquad i = 1, 2, \ldots, m$$

For our one-dimensional rod, Eq. (15) can be substituted into this formulation to give

$$\int_{x_1}^{x_2} \left[\frac{d^2\tilde{T}}{dx^2} + f(x) \right] N_i \, dx \qquad i = 1, 2$$

which can be reexpressed as

$$\int_{x_1}^{x_2} \frac{d^2\tilde{T}}{dx^2} N_i(x) \, dx = -\int_{x_1}^{x_2} f(x) N_i(x) \, dx \qquad i = 1, 2 \qquad (17)$$

At this point, a number of mathematical manipulations will be applied to simplify and evaluate Eq. (17). Among the most important is the simplification of the left-hand side using integration by parts. Recall from calculus that this operation can be expressed generally as

$$\int_a^b u \, dv = uv\big|_a^b - \int_a^b v \, du$$

If *u* and v are chosen properly, the new integral on the right-hand side will be easier to evaluate than the original one on the left-hand side. This can be done for the term on the left-hand side of Eq. *(* 17) by choosing $N_i(x)$ as *u* and $(d^2T/dx^2)dx$ as dv to yield

$$\int_{x_1}^{x_2} N_i(x) \frac{d^2\tilde{T}}{dx^2} \, dx = N_i(x) \frac{d\tilde{T}}{dx}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d\tilde{T}}{dx} \frac{dN_i}{dx} \, dx \qquad i = 1, 2 \qquad (18)$$

Next, we can evaluate the individual terms that we have created in Eq. (18). For i = 1, the first term on the right-hand side of Eq. *(*18) can be evaluated as

$$N_1(x) \frac{d\tilde{T}}{dx}\bigg|_{x_1}^{x_2} = N_1(x_2) \frac{d\tilde{T}(x_2)}{dx} - N_1(x_1) \frac{d\tilde{T}(x_1)}{dx}$$

However, recall from Fig. *2* that $N_1(x_2) = 0$ and $N_l(x_l) = 1$, and therefore,

$$N_1(x) \frac{d\tilde{T}}{dx}\bigg|^{x_2} = -\frac{d\tilde{T}(x_1)}{dx} \qquad (19)$$

Similarly, for i = *2*,

$$N_2(x) \frac{d\tilde{T}}{dx}\bigg|_{x_1}^{x_2} = \frac{d\tilde{T}(x_2)}{dx} \qquad (20)$$

Thus, the first term on the right-hand side of Eq. *(18)* represents the natural boundary conditions at the ends of the elements.
Now, before proceeding let us regroup by substituting our results back into the original equation. Substituting Eqs. *(18)* through *(20)* into Eq. *(17)* and rearranging gives for i = 1,

$$\int_{x_1}^{x_2} \frac{d\tilde{T}}{dx} \frac{dN_1}{dx} \, dx = -\frac{d\tilde{T}(x_1)}{dx} + \int_{x_1}^{x_2} f(x) N_1(x) \, dx \qquad (21)$$

and for i = *2*,