$$\int_{x_1}^{x_2} \frac{d\tilde{T}}{dx}\frac{dN_2}{dx}\,dx = \frac{d\tilde{T}(x_2)}{dx} + \int_{x_1}^{x_2} f(x)N_2(x)\,dx \qquad (22)$$

Notice that the integration by parts has led to two important outcomes. First, it has incorporated the boundary conditions directly into the element equations. Second, it has lowered the highest-order evaluation from a second to a first derivative. This latter outcome yields the significant result that the approximation functions need to preserve continuity of value but not slope at the nodes.

Also notice that we can now begin to ascribe some physical significance to the individual terms we have derived. On the right-hand side of each equation, the first term represents one of the element's boundary conditions and the second is the effect of the system's forcing function-in the present case, the heat source *f(x)*. As will now become evident, the left-hand side embodies the internal mechanisms that govern the element's temperature distribution. That is, in terms of the finite-element method, the left-hand side will become the element property matrix.

To see this, let us concentrate on the terms on the left-hand side. For i = *1,* the term is

$$\int_{x_1}^{x_2} \frac{d\tilde{T}}{dx}\frac{dN_1}{dx}\,dx \qquad (23)$$

Recall from Sec. *1.2* that the linear nature of the shape function makes differentiation and integration simple. Substituting Eqs. (*6)* and *(7)* into Eq. *(23)* gives

$$\int_{x_2}^{x_1} \frac{T_1 - T_2}{(x_2 - x_1)^2}\,dx = \frac{1}{x_1 - x_2}(T_1 - T_2) \qquad (24)$$

Similar substitutions for i = *2* [Eq. *(22)]* yield

$$\int_{x_2}^{x_1} \frac{-T_1 + T_2}{(x_2 - x_1)^2}\,dx = \frac{1}{x_1 - x_2}(-T_1 + T_2) \qquad (25)$$

Comparison with Eq. *(14)* shows that these are similar to the relationships that were developed with the direct method using Fourier's law. This can be made even clearer by re-expressing Eqs. ***(24)*** and ***(25)*** in matrix form as

$$\frac{1}{x_2 - x_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

Substituting this result into Eqs. ***(21)*** and ***(22)*** and expressing the result in matrix form gives the final version of the element equations

$$\underbrace{\frac{1}{x_2 - x_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\text{Element stiffness matrix}}\{T\} = \underbrace{\begin{Bmatrix} -\dfrac{dT(x_1)}{dx} \\ \dfrac{dT(x_2)}{dx} \end{Bmatrix}}_{\text{Boundary condition}} + \underbrace{\begin{Bmatrix} \displaystyle\int_{x_1}^{x_2} f(x)N_1(x)\,dx \\ \displaystyle\int_{x_1}^{x_2} f(x)N_2(x)\,dx \end{Bmatrix}}_{\text{External effects}} \qquad (26)$$

3. The Method of Weighted Residuals (MWR) for solving ODEs

The MWR will be presented as; Suppose we have a linear differential Operator D acting on a function u to produce a function p.

$$D\big(u(x)\big) = p(x)$$

We wish to approximate u by a function $\tilde{u}$, which is a linear combination of basis functions chosen from a linearly independent set. That is,

$$u \cong \tilde{u} = \sum_{i=1}^{n} a_i\varphi_i \qquad (3\text{-}1)$$

Now, when substituted into the differential operator, D, the result of the operations is not, in general, $p(x)$. Hence an error or residual will exist:

$$E(x) = R(x) = D\big(\tilde{u}(x)\big) - p(x) \neq 0$$

The notation in the MWR is to force the residual to zero in some average sense over the domain. That is;

$$\int_x R(x)\ W_i dx = 0\ ; \qquad i=1,2,\ldots,n \qquad (3\text{-}2)$$

where, the No. of *weight functions* W_i is exactly equal the No. of unknown constants a_i in $\tilde{u}$. The result is a set of *n* algebraic equations for the unknown constants a_i. MWR are used and classified according to choices of the W_i*'s*, these methods are:-

3.1.Point collocation method:

In this method, the weighting functions are taken from the family of *Dirac Delta* functions in the domain. That is $W_i(x) = \delta(x - x_i)$ $\rightarrow$ { Unit impulse }

The *Dirac* δ function has the properly that;

$$\delta\ (x - x_i) = \begin{cases} 1 & x = x_i \\ 0 & otherwise \end{cases} \qquad (\text{ it means Unit impulse })$$

Hence the integration of the weighted residual statement results in the forcing of the residual to *zero* at specific points in the domain. That is

$$\int_0^1 R(x_i)\ \delta(x - x_i)dx = \int_0^1 R(x_i)\ 1\ dx$$

$$= R(x_i).x|_0^1 = R(x_i) - 0 \qquad \text{forced to} \xrightarrow{yields} 0$$

Hence; $R(x_i) = 0 \equiv D\big(u(x)\big) - p(x)$

Example (3-1): Find $u(x)$ that satisfies

$\frac{d^2u}{dx^2} + u = 1;$ & B.Cs. *u(0) =1 & u(1) =0* ; it's *D operator & p(x)* are

$D\big(u(x)\big) = \left(\frac{d^2}{dx^2} + 1\right) u(x)\ ;\ \&\ \ p(x) = 1$

Solution: Let the approximation function $\tilde{u}(x) = a_0 + a_1x + a_2x^2$

Application of the BCs reveals; $\tilde{u}(0) = 1 = a_0$

$\tilde{u}(1) = 0 = 1 + a_1 + a_2$ or $a_1 = -(1 + a_2)$

and $\tilde{u}(x)$ *which* satisfies BC's is $\tilde{u}(x) = 1 - (1 + a_2)x + a_2x^2$

$= 1 - x - a_2x + a_2x^2 = 1 - x + a_2(x^2 - x)$

To find $R(x)$, we need the 2nd derivative of $\tilde{u}(x)$

$$\frac{d\tilde{u}(x)}{dx} = -1 - a_2 + 2a_2x$$

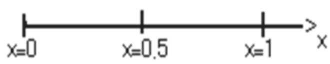
$$\frac{d^2\tilde{u}(x)}{dx^2} = 2a_2$$

Hence; $R(x) = D\big(\tilde{u}(x)\big) - p(x) = \frac{d^2\tilde{u}(x)}{dx^2} + \tilde{u} - 1 = 0$

$= 2a_2 + [1 - x + a_2x^2 - a_2x] - 1 = 2a_2 + 1 - x + a_2x^2 - a_2x - 1 = -x + a_2(x^2 - x + 2) = 0$

Since there is only one unknown (a_2), only one collocation point is needed. We choose (arbitrarily, but from symmetry considerations) the collocation point *x=0.5*;

$R(x) = -0.5 + a_2\,(0.5^2 - 0.5 + 2) = 0\ 0$



$a_2\,(0.25 - 0.5 + 2) = 0.5 \xrightarrow{yields} a_2 = \frac{2}{7} = 0.2857143\ ;\ a_1 = -(1 + 0.2857143) = -1.286$

Hence ; $\tilde{u}(x)$ *which* satisfies BC's is $\boldsymbol{\tilde{u}(x) = 1 - 1.286x + 0.286x^2}$

Example (3-2): Find $u(x)$ that satisfies

$\frac{d^2u}{dx^2} + u = -x;$ & B.Cs. *u(0) =0 & u(1) =0* ;

Solution: it's *D operator & p(x)* are

$D\big(u(x)\big) = \left(\frac{d^2}{dx^2} + 1\right) u(x)\ ;\ \&\ \ p(x) = -x$

Let the approximation function $\tilde{u}(x) = x(1 - x)(a_1 + a_2x) + \cdots = a_1x - a_1x^2 + a_2x^2 - a_2x^3$

Application of the BCs reveals; $\tilde{u}(0) = 0 \neq a_0$

$\tilde{u}(1) = 0 = 1(1 - 1)(a_1 + a_2 1) \xrightarrow{yields} a_2 = -a_1$

To find $R(x)$, we need the 2nd derivative of $\tilde{u}(x)$

$$\frac{d\tilde{u}(x)}{dx} = a_1 - 2a_1x + 2a_2x - 3a_2x^2$$

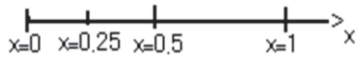
putting in $R(x) = D\big(\tilde{u}(x)\big) - p(x) = 0$

$$= -2a_1 + 2a_2 - 6a_2x + [a_1x - a_1x^2 + a_2x^2 - a_2x^3] + x = 0$$

$$= x + a_1(-2 + x - x^2) + a_2(2 - 6x + x^2 - x^3) = 0$$

Since the Unknowns are two { i.e. a_1 & a_2 }; two collocation points are needed; these are (0.5, 0.25)

At *x=0.5*,



$R(0.5) = 0.5 + a_1(-2 + 0.5 - 0.5^2) + a_2(2 - 6(0.5) + 0.5^2 - 0.5^3) = 0$

$= 0.5 + a_1(-1.75) + a_2(0.875) = 0$ (Ex. 3-2-Eq1)

At *x=0.25*,

$R(0.25) = 0.25 + a_1(-2 + 0.25 - 0.25^2) + a_2(2 - 6(0.25) + 0.25^2 - 0.25^3) = 0$

$= 0.25 + a_1(-1.8125) + a_2(0.546875) = 0$ (Ex. 3-2-Eq2)

Solve Eqs (Eq1 & Eq2) simultaneously we get;

$a_1 = 0.1935484 \cong 0.194$ & $a_2 = 0.1843318 \cong 0.184$

Hence ; $\tilde{u}(x)$ *which* satisfies BC's is $\boldsymbol{\tilde{u}(x) = x(1-x)(0.194 + 0.184x)}$

3.2. Galerkin Method

Recalling Eq. (3-2) $\int_0^1 R(x)\ W_i dx = 0$; *i=1, 2, …., n*

In this method, the weight function W_i is the derivative of the approximating function $\tilde{u}(x)$ with respect to the unknown coefficient a_i .

Example (3-3)

For Example (3-1) $\frac{d^2u}{dx^2} + u = 1$; & B.Cs. $u(0) = 1$ & $u(1) = 0$; $\tilde{u}(x) = 1 - x + a_2(x^2 - x)$

$W_1 = \frac{d\tilde{u}}{da_2} = x^2 - x$

Hence; $\int_0^1 R(x)\ W_1 dx = 0 = \int_0^1 (x^2 - x)\,[-x\ +\ a2\ (\ x2 -\ x\ + 2\ )]dx = 0$

$$= \int_0^1 [-x^3\ +\ a_2\,x^4 - 2a_2x^3 + 3a_2x^2 - 2a_2x + x^2]dx = 0$$

$\rightarrow \rightarrow a_2 = \frac{1}{12} * \frac{10}{3} = \frac{5}{18} = 0.2778$

Hence ; $\tilde{u}(x)$ *which* satisfies BC's is $\underline{\boldsymbol{\tilde{u}(x) = 1 - 1.2778x + 0.2778x^2}}$

Example (3-4)

For Example (3-2); $\tilde{u}(x) = x(1-x)(a_1 + a_2x) + \cdots = a_1x - a_1x^2 + a_2x^2 - a_2x^3$

which gives $R(x) = x + a_1(-2 + x - x^2) + a_2(2 - 6x + x^2 - x^3) = 0$

Solution: Recalling Eq. (3-2) $\int_0^1 R(x)\ W_i dx = 0$

and rearranging ; $\tilde{u}(x) = a_1(x - x^2) + a_2(x^2 - x^3)$

then finding $W_1 = \frac{d\tilde{u}}{da_1} = x - x^2$; $W_2 = \frac{d\tilde{u}}{da_2} = x^2 - x^3$

Eq. (3-2) becomes $\int_0^1 R(x)\ W_1W_2 dx =$

$= \int_0^1 (x - x^2)(x^2 - x^3)\ [x\ +\ a_1(-2 + x - x^2) + a_2(2 - 6x + x^2 - x^3)]\ \ dx =$

In Matrix Form

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$b_{11} = \int_0^1 (-2 + x - x^2)(x - x^2)dx$ (e1)

$b_{22} = \int_0^1 (2 - 6x + x^2 - x^3)(x^2 - x^3)dx$ (e2)

$b_{12} = \int_0^1 (2 - 6x + x^2 - x^3)(x - x^2)dx$ (e3)

$b_{21} = \int_0^1 (-2 + x - x^2)(x^2 - x^3)dx$ (e4)

$c_1 = \int_0^1 x(x - x^2)dx$ (e5)

$c_2 = \int_0^1 x(x^2 - x^3)dx$ (e6)

Solving Eqs. (e1,…, e4) to find b_{11}, ….., b_{21}; take e1

$$b_{11} = \int_0^1 (-2x + x^2 - x^3 + 2x^2 - x^3 + x^4)dx$$

$$b_{11} = \left[-x^2 + \frac{x^3}{3} - \frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\right]\Big|_0^1$$

$$b_{11} = \left[-1 + \frac{1}{3} - \frac{1}{4} + \frac{2}{3} - \frac{1}{4} + \frac{1}{5}\right] - [0] = -\frac{3}{10}$$

So by integrating Eqs.(e2, e3, e4), we finding ; $b_{22} = -\frac{13}{105}$; $b_{12} = -\frac{3}{20}$; $b_{21} = -\frac{3}{20}$;

Hence ; $[b_{ij}] = \begin{bmatrix} -\frac{3}{10} & -\frac{3}{20} \\ -\frac{3}{20} & -\frac{13}{105} \end{bmatrix}$

solving Eqs (e5,e6) to finding $c_1 = \frac{1}{12}$; $c_2 = \frac{1}{20}$;

Hence the Eqs. system is

$\begin{bmatrix} -\frac{3}{10} & -\frac{3}{20} \\ -\frac{3}{20} & -\frac{13}{105} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{12} \\ \frac{1}{20} \end{bmatrix}$, solving this system to find $a_1 = -\frac{3}{9}$; $a_2 = \frac{1}{9}$

Hence ; $\tilde{u}(x)$ *which* satisfies BC's is $\boldsymbol{\tilde{u}(x) = -\frac{3}{9}x + \frac{4}{9}x^2 - \frac{1}{9}x^3}$

3.3.Least Squares Method

Recalling Eq. (3-2) $\int_0^1 R(x)\ W_i dx = 0$; *i=1, 2, …., n*

In this method, the weight function W_i is the derivative of $R(x)$ with respect to the unknown coefficient a_i .

Example (3-5)

For Example (3-1)

$\frac{d^2u}{dx^2} + u = 1$; $p(x) = 1$; & B.Cs. $u(0) = 1$ & $u(1) = 0$; $\tilde{u}(x) = 1 - x + a_2(x^2 - x)$

$$R(x) = D(\tilde{u}(x)) - p(x) = \frac{d^2\tilde{u}(x)}{dx^2} + \tilde{u} - p(x)$$

$$= -x + a_2(x^2 - x + 2)$$

$$\to\to W_1 = \frac{dR}{da_2} = x^2 - x + 2$$

The weighted residual statement becomes $\int_0^1 W_1 R(x)\ dx = 0$

$$\int_0^1 (x^2 - x + 2)[2a_2 - x + a_2x^2 - a2x]\ dx = 0$$

$$\int_0^1 [-x^3 + a_2x^4 - a_2x^3 + 2a_2x^2 + x^2 - a_2x^3 + a_2x^2 - 2a_2x - 2x + 2a_2x^2 - 2a_2x + 4a_2]dx$$

$$\int_0^1 [-x^3 + x^2 - 2x + a_2x^4 - 2a_2x^3 + 5a_2x^2 - 4a_2x + 4a_2]dx$$

$$= \left[-\frac{x^4}{4} + \frac{x^3}{3} - \frac{2x^2}{2} + \frac{a_2x^5}{5} - \frac{2a_2x^4}{4} + \frac{5a_2x^3}{3} - \frac{4a_2x^2}{2} + 4a_2x\right]\Big|_0^1$$

$$= \left[-\frac{1}{4} + \frac{1}{3} - 1 + \frac{a_2}{5} - \frac{a_2}{2} + \frac{5a_2}{3} - 2a_2 + 4a_2\right] - [0] = \left[-\frac{11}{12} + \frac{101}{30}a_2\right] = 0$$

$$\to\to a_2 = \frac{55}{202} = 0.272277$$

Hence ; $\tilde{u}(x)$ *which* satisfies BC's is $\boldsymbol{\tilde{u}(x) = 1 - 1.272277x + 0.272277x^2}$

H.W.: Resolve Example (3-2) by L. S. Method

Final Results are

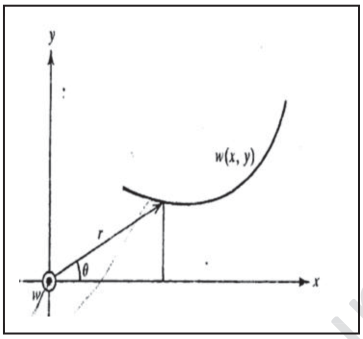
$\begin{bmatrix} 3.36667 & 1.68333 \\ 1.68333 & 3.7428 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.91667 \\ 6.55000 \end{bmatrix}$ **; and Compare all results of all method for both Examples.**

Lecture 8: TRANSFORMATION

1- Transformation From Cartesian To Polar Coordinates

The Function *W(x,y)* can be defined at any point *(x,y)* in polar form as :

$\cos\theta = \frac{x}{r}$; $\sin\theta = \frac{y}{r}$; and $\tan\theta = \frac{y}{x}$



In addition,

$r^2 = x^2 + y^2$...... (1) and $\theta = \tan^{-1}\frac{y}{x}$ ....(2)

Taking 1st Partial derivatives of W with respect to:

x: $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x}$ ....(3)

y: $\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial y}$ ....(4)

From *eq(1)*

$2r\,\partial r = 2x\,\partial x + 2y\,\partial y$

So

$2r\frac{\partial r}{\partial x} = 2x \quad \rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$

and

$2r\frac{\partial r}{\partial y} = 2y \quad \rightarrow \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$

From *eq(2)* the 1st Partial derivatives of θ with respect to x is :

$\frac{\partial\theta}{\partial x} = \frac{-y/x^2}{1+(y/x)^2} = \frac{-y}{x^2+y^2} = -\frac{1}{r}\sin\theta$ ◀==========================

NOTE

$r^2 = x^2 + y^2$

$\rightarrow \frac{1}{r}\frac{1}{r} = \frac{1}{x^2+y^2}\frac{1}{r^2} = \frac{1}{x^2+y^2}$

$\frac{-1}{r}\frac{y}{r} = \frac{-y}{x^2+y^2} = \frac{-1}{r}(\sin\theta)$

And the 1st Partial derivatives of θ with respect to y is :

$$\frac{\partial\theta}{\partial y} = \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2} = \frac{1}{r}\cos\theta$$

Substituting these relations into *eq(3)* permits

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\,(\cos\theta) + \frac{\partial w}{\partial \theta}\left(-\frac{1}{r}\sin\theta\right) \dots\dots\dots\dots\dots\dots\dots (5)$$

And

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\,(\sin\theta) + \frac{\partial w}{\partial \theta}\left(\frac{1}{r}\cos\theta\right) \dots\dots\dots\dots\dots\dots\dots\dots (6)$$

The 2nd Partial derivatives are

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}(Q)$$

$$= \frac{\partial Q}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial Q}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \cos\theta\frac{\partial Q}{\partial r} + \left(-\frac{1}{r}\sin\theta\right)\frac{\partial Q}{\partial \theta}$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\cos\theta - \frac{1}{r}\sin\theta\frac{\partial w}{\partial \theta}\right) - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial w}{\partial r}\cos\theta - \frac{1}{r}\sin\theta\frac{\partial w}{\partial \theta}\right)$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial r^2}\cos^2\theta + \frac{\partial^2 w}{\partial \theta^2}\frac{\sin^2\theta}{r^2} - 2\frac{\partial^2 w}{\partial\theta\partial r}\frac{\sin\theta\cos\theta}{r} + \frac{\partial w}{\partial r}\frac{\sin^2\theta}{r} + 2\frac{\partial w}{\partial \theta}\frac{\sin\theta\cos\theta}{r^2} \dots\dots\dots\dots (7)$$

And

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2}\sin^2\theta + \frac{\partial^2 w}{\partial \theta^2}\frac{\cos^2\theta}{r^2} + 2\frac{\partial^2 w}{\partial\theta\partial r}\frac{\sin\theta\cos\theta}{r} + \frac{\partial w}{\partial r}\frac{\cos^2\theta}{r} - 2\frac{\partial w}{\partial \theta}\frac{\sin\theta\cos\theta}{r^2} \dots\dots\dots\dots (8)$$

The mixed P.D. is determined by using;

$$\frac{\partial^2 w}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial R}{\partial x}$$

Therefore

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial R}{\partial x} = \frac{\partial R}{\partial r}\,\frac{\partial r}{\partial x} + \frac{\partial R}{\partial \theta}\,\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial R}{\partial r}\cos\theta + \frac{\partial R}{\partial \theta}\left(-\frac{1}{r}\sin\theta\right)$$

$$= \cos\theta\,\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\,\sin\theta + \frac{\partial w}{\partial \theta}\,\frac{1}{r}\cos\theta\right) - \frac{1}{r}\sin\theta\,\frac{\partial}{\partial \theta}\left(\frac{\partial w}{\partial r}\,\sin\theta + \frac{\partial w}{\partial \theta}\,\frac{1}{r}\cos\theta\right)$$

$$= \cos\theta\left(\sin\theta\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\cos\theta\frac{\partial^2 w}{\partial r\partial\theta} - \frac{\cos\theta}{r^2}\frac{\partial w}{\partial\theta}\right) - \frac{1}{r}\sin\theta\left(\sin\theta\frac{\partial^2 w}{\partial r\partial\theta} + \cos\theta\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial\theta^2}\frac{1}{r}\cos\theta - \frac{\partial w}{\partial\theta}\frac{1}{r}\sin\theta\right)$$

$$\frac{\partial^2 w}{\partial x\partial y} = \sin\theta\cos\theta\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}(\cos^2\theta - \sin^2\theta)\frac{\partial^2 w}{\partial r\partial\theta} + \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial^2 w}{\partial\theta^2}$$
$$+ \frac{1}{r^2}(\sin^2\theta - \cos^2\theta)\frac{\partial w}{\partial\theta} - \frac{1}{r}\sin\theta\cos\theta\frac{\partial w}{\partial r} \ldots\ldots\ldots\ldots\ldots. (9)$$

==

Example:

Transform the well known Laplace equation of steady state fluid flow from the Cartesian coordinate system to its equivalent polar form.

Solution:

Laplace equation is $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$

Put *w* instead of *h* we get $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$

Using *eqs (7 & 8)*

$$\frac{\partial^2 w}{\partial r^2}\cos^2\theta + \frac{\partial^2 w}{\partial\theta^2}\frac{\sin^2\theta}{r^2} - 2\frac{\partial^2 w}{\partial\theta\partial r}\frac{\sin\theta\cos\theta}{r} + \frac{\partial w}{\partial r}\frac{\sin^2\theta}{r} + 2\frac{\partial w}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2}$$
$$+ \frac{\partial^2 w}{\partial r^2}\sin^2\theta + \frac{\partial^2 w}{\partial\theta^2}\frac{\cos^2\theta}{r^2} + 2\frac{\partial^2 w}{\partial\theta\partial r}\frac{\sin\theta\cos\theta}{r} + \frac{\partial w}{\partial r}\frac{\cos^2\theta}{r} - 2\frac{\partial w}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2} = 0$$

$$\frac{\partial^2 w}{\partial r^2}\cos^2\theta + \frac{\partial^2 w}{\partial\theta^2}\frac{\sin^2\theta}{r^2} \cancel{- 2\frac{\partial^2 w}{\partial\theta\partial r}\frac{\sin\theta\cos\theta}{r}} + \frac{\partial w}{\partial r}\frac{\sin^2\theta}{r} \cancel{+ 2\frac{\partial w}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2}}$$
$$+ \frac{\partial^2 w}{\partial r^2}\sin^2\theta + \frac{\partial^2 w}{\partial\theta^2}\frac{\cos^2\theta}{r^2} \cancel{+ 2\frac{\partial^2 w}{\partial\theta\partial r}\frac{\sin\theta\cos\theta}{r}} + \frac{\partial w}{\partial r}\frac{\cos^2\theta}{r} \cancel{- 2\frac{\partial w}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2}} = 0$$

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{\partial^2 h}{\partial r^2}(\cos^2\theta + \sin^2\theta) + \frac{\partial^2 h}{\partial\theta^2}\left(\frac{\sin^2\theta + \cos^2\theta}{r^2}\right) + \frac{\partial w}{\partial r}\left(\frac{\sin^2\theta + \cos^2\theta}{r}\right)$$
$$= \frac{\partial^2 h}{\partial r^2}(1) + \frac{\partial^2 h}{\partial\theta^2}\left(\frac{1}{r^2}\right) + \frac{\partial h}{\partial r}\left(\frac{1}{r}\right)\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{\partial^2 h}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 h}{\partial\theta^2} + \frac{1}{r}\frac{\partial h}{\partial r}\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

HomeWork: Adapt eq (10) to be sutable to determine the steady state temperature distribution at the interior nodes of the Fig. (1).

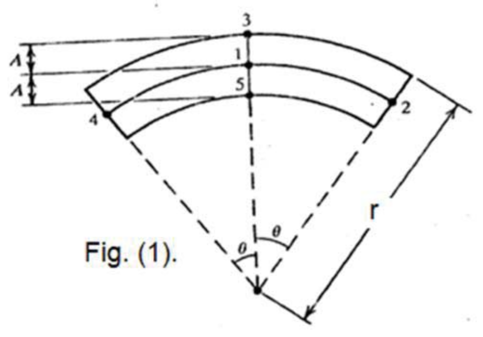


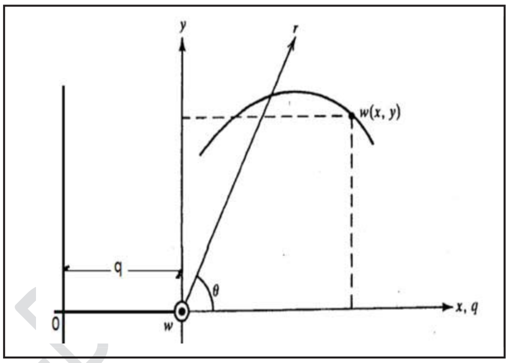
Fig. (1).

The answer is

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=-\left(\frac{2}{A^2}+\frac{2}{r^2\theta^2}\right)T_1+\left(\frac{1}{r^2\theta^2}\right)T_2+\left(\frac{1}{A^2}+\frac{1}{2rA}\right)T_3+\left(\frac{1}{r^2\theta^2}\right)T_4+\left(\frac{1}{A^2}-\frac{1}{2rA}\right)T_5$$

2- <u>Transformation From Cartesian To Skewed Coordinates</u>

Fluid flow, temperature, and plate deflection are engineering examples of such occurrences. Using the skewed coordinate system. Consider the function *W(x,y)* in the Fig.



$x = q + r\cos\theta$ ………………(11)

$y = r\sin\theta$ ……………………....(12)

Determining the Partial Derivatives in the skewed coordinate System, that is;

$$\frac{\partial w}{\partial q}=\frac{\partial w}{\partial x}\ \frac{\partial x}{\partial q}+\frac{\partial w}{\partial y}\ \frac{\partial y}{\partial q}\ \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\ \frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\ \frac{\partial y}{\partial r}\ \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

From *eqs (11 &12)*, we have;

$$\left.\begin{aligned}&\frac{\partial x}{\partial q}=1\ \ \ldots\ldots\ldots\ldots(15)\ ;\ \ \frac{\partial x}{\partial r}=\cos\theta\ \ldots\ldots\ldots\ldots(16)\\&\frac{\partial y}{\partial q}=0\ \ \ldots\ldots\ldots\ldots(17)\ ;\ \ \frac{\partial y}{\partial r}=\sin\theta\ \ldots\ldots\ldots\ldots(18)\end{aligned}\right\}$$

***Note** that θ is constant in this case*

Substituting *eqs (15 & 17)* into *equation (13)* gives;

$$\frac{\partial w}{\partial q}=\frac{\partial w}{\partial x}\ 1+\frac{\partial w}{\partial y}\ 0\ \ \ldots\ldots\ldots\ldots(19)$$

Taking 2nd P.D. of *eq (19)* with respect to *q*

$$\frac{\partial^2 w}{\partial q^2}=\frac{\partial}{\partial q}\left(\frac{\partial w}{\partial q}\right)=\frac{\partial}{\partial q}\left(\frac{\partial w}{\partial x}\right)=\frac{\partial^2 w}{\partial x^2}\ \ \ldots\ldots\ldots(20)$$

{ **Note in eq 15**; $\partial x = \partial q$

Now substituting *eqs (16 & 18)* into *eq (14)* gives;

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\ \cos\theta+\frac{\partial w}{\partial y}\ \sin\theta\ \ \ldots\ldots\ldots\ldots\ldots(21)$$

Taking 2nd P.D. of *eq (21)* with respect to *r*

$$\frac{\partial^2 w}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right)=\frac{\partial Q}{\partial r}=\frac{\partial Q}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial Q}{\partial y}\frac{\partial y}{\partial r}$$

$$=\frac{\partial Q}{\partial x}\cos\theta+\frac{\partial Q}{\partial y}\sin\theta$$

$$=\cos\theta\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\ \cos\theta+\frac{\partial w}{\partial y}\ \sin\theta\right)+\sin\theta\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\ \cos\theta+\frac{\partial w}{\partial y}\ \sin\theta\right)$$

$$=\cos^2\theta\frac{\partial^2 w}{\partial x^2}+\ \sin\theta\cos\theta\frac{\partial^2 w}{\partial x\partial y}\ +\ \sin\theta\cos\theta\frac{\partial^2 w}{\partial x\partial y}\ +\sin^2\theta\frac{\partial^2 w}{\partial y^2}$$

$$\frac{\partial^2 w}{\partial r^2}=\cos^2\theta\frac{\partial^2 w}{\partial x^2}+2\ \sin\theta\cos\theta\frac{\partial^2 w}{\partial x\partial y}\ +\sin^2\theta\frac{\partial^2 w}{\partial y^2}\ \ \ldots\ldots\ldots\ldots(22)$$

We need to define the mixed 2nd P.D.

$$\frac{\partial^2 w}{\partial q\partial r}=\frac{\partial}{\partial q}\left(\frac{\partial w}{\partial r}\right)=\frac{\partial R}{\partial q}$$

$$=\frac{\partial R}{\partial x}\frac{\partial x}{\partial q}+\frac{\partial R}{\partial y}\frac{\partial y}{\partial q}$$

or by using *eqs (15 & 17)* it equals;

$$\frac{\partial^2 w}{\partial q\partial r}=\frac{\partial R}{\partial x}1+\frac{\partial R}{\partial y}0=\frac{\partial R}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\ \cos\theta+\frac{\partial w}{\partial y}\ \sin\theta\right)$$

$$\frac{\partial^2 w}{\partial q\partial r}=\cos\theta\frac{\partial^2 w}{\partial x^2}+\sin\theta\frac{\partial^2 w}{\partial x\partial y}\ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

Eqs (19 &21), can expressed in matrix form as;

$$\begin{Bmatrix} \frac{\partial w}{\partial q} \\ \frac{\partial w}{\partial r} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{Bmatrix} \quad \text{......................}(24)$$

Eq (24) can be easily used to solve 1st P.D. in Cartesian System.

The 2nd P.D. in the Cartesian system can be solved by using *eqs (20, 22, &23)* as;

$$\begin{Bmatrix} \frac{\partial^2 w}{\partial q^2} \\ \frac{\partial^2 w}{\partial q \partial r} \\ \frac{\partial^2 w}{\partial r^2} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \cos\theta & \sin\theta & 0 \\ \cos^2\theta & 2\sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial x \partial y} \\ \frac{\partial^2 w}{\partial y^2} \end{Bmatrix} \quad \text{.............}(25)$$

Rewriting eq (25) in compact matrix form, we have;

$$\{W_s\} = [\theta]\{W_c\}$$

Solving for the unknown vector involving the Partial in Cartesian coordinates gives;

$$\{W_c\} = [\theta]^{-1}\{W_s\} \quad \text{................}(26)$$

Example: Transform the Laplace equation of steady state fluid flow into the skewed coordinate system using θ= 45°.

Solution: using the 2- dimensional form we have

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Substituting θ= 45° into the matrix of equation (25) gives

$$\{W_s\} = \begin{bmatrix} 1 & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/2 & 1 & 1/2 \end{bmatrix} \{W_c\}$$

Solving by the inverse procedure gives

$$\{W_c\} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & \sqrt{2} & 0 \\ 1 & -2\sqrt{2} & 2 \end{bmatrix} \{W_s\}$$

Which equivalent to

$$\begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial x \partial y} \\ \frac{\partial^2 w}{\partial y^2} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & \sqrt{2} & 0 \\ 1 & -2\sqrt{2} & 2 \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w}{\partial q^2} \\ \frac{\partial^2 w}{\partial q \partial r} \\ \frac{\partial^2 w}{\partial r^2} \end{Bmatrix}$$

Thus

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial q^2} \quad \text{................}(27)$$

$$\frac{\partial^2 w}{\partial x \partial y} = -\frac{\partial^2 w}{\partial q^2} + \sqrt{2}\frac{\partial^2 w}{\partial q \partial r} \quad \text{......}(28)$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial q^2} - 2\sqrt{2}\frac{\partial^2 w}{\partial q \partial r} + 2\frac{\partial^2 w}{\partial r^2} \quad \text{.......}(29)$$

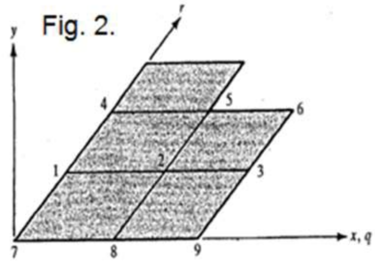
Note that eq (28) is not needed in this case (i.e. since Laplace eq $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$). Hence adding equations (27, & 29) gives;

$$\frac{\partial^2 w}{\partial q^2}+\frac{\partial^2 w}{\partial q^2}-2\sqrt{2}\,\frac{\partial^2 w}{\partial q\partial r}+2\frac{\partial^2 w}{\partial r^2}=0$$

Simplifying, we obtain

$$\frac{\partial^2 w}{\partial q^2}-\sqrt{2}\,\frac{\partial^2 w}{\partial q\partial r}+\frac{\partial^2 w}{\partial r^2}=0 \quad \text{.............(30)}$$ which is the Laplace eq in skewed coordinates with $\theta= 45^o$.

Fig. 2.



HomeWork:

Transform the Laplace equation of steady state temperature distribution in skewed coordinate system (q, r) of the Fig. (2), using $\theta= 60^o$.

Then find steady state temperature distribution at node 2 assuming $\Delta q=\Delta r=1$.

The Answer is

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{4}{3}(T_1-2T_2+T_3)+\frac{4}{3}(T_5-2T_2+T_8)-\frac{4}{12}(-T_4+T_6+T_7-T_9)$$

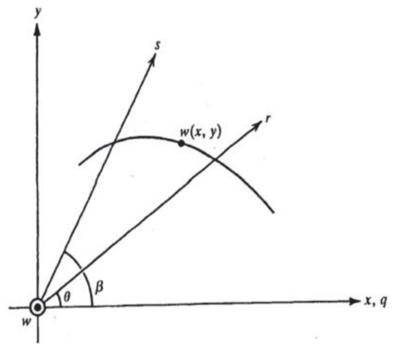
3- Transformation From Cartesian To Triangular Coordinates

Consider the function *W(x,y)* in the Fig.

Triangular elements represent the building block for many F.E. Techniques, are;

$$x=q+r\cos\theta+s\cos\beta \quad \text{..................(31)}$$

$$y=r\sin\theta+s\sin\beta \quad \text{.........................(32)}$$



Note that x & y are expressed in terms of three variables. Therefore

$$\frac{\partial w}{\partial q}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial q}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial q} \quad \text{..........(33)}$$

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}\text{..........(34)}; \quad \frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}\text{........(35)}$$

From *eqs (31 &32)*, we have;

$$\frac{\partial x}{\partial q}=1 \quad ; \quad \frac{\partial y}{\partial q}=0 \quad ; \quad \frac{\partial x}{\partial r}=\cos\theta \quad ; \quad \frac{\partial y}{\partial r}=\sin\theta$$

$$\frac{\partial x}{\partial s}=\cos\beta \quad ; \quad \frac{\partial y}{\partial s}=\sin\beta$$

Substituting *them* into *equations* (*33, 34, & 35*) to gives;

$$\frac{\partial w}{\partial q}=\frac{\partial w}{\partial x}\;1+\frac{\partial w}{\partial y}\;0=\frac{\partial w}{\partial x}\;; \qquad \frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\;\cos\theta+\frac{\partial w}{\partial y}\;\sin\theta\;; \qquad \frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\;\cos\beta+\frac{\partial w}{\partial y}\;\sin\beta$$

The 2nd P. D. are computed to be given as;

$$\frac{\partial^2 w}{\partial q^2}=\frac{\partial}{\partial q}\left(\frac{\partial w}{\partial q}\right)=\frac{\partial}{\partial q}\left(\frac{\partial w}{\partial x}\right)=\frac{\partial^2 w}{\partial x^2}$$ { **Note that**; $\partial x=\partial q$ as in above

$$\frac{\partial^2 w}{\partial r^2}=\cos^2\theta\frac{\partial^2 w}{\partial x^2}+2\;\sin\theta\cos\theta\frac{\partial^2 w}{\partial x\partial y}\;+\sin^2\theta\frac{\partial^2 w}{\partial y^2}$$

and

$$\frac{\partial^2 w}{\partial s^2}=\cos^2\beta\frac{\partial^2 w}{\partial x^2}+2\;\sin\beta\cos\beta\frac{\partial^2 w}{\partial x\partial y}\;+\sin^2\beta\frac{\partial^2 w}{\partial y^2}$$

These equations can be readily expressed in a matrix form as;

$$\begin{Bmatrix} \frac{\partial^2 w}{\partial q^2} \\ \frac{\partial^2 w}{\partial r^2} \\ \frac{\partial^2 w}{\partial s^2} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ cos^2\theta & 2sin\,\theta\,cos\,\theta & sin^2\theta \\ cos^2\beta & 2sin\,\beta\,cos\,\beta & sin^2\beta \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial x \partial y} \\ \frac{\partial^2 w}{\partial y^2} \end{Bmatrix} \quad \text{.............(36a)}$$

Or in compact matrix form;

$\{W_t\} = [A]\{W_c\}$ ....(36b)

Eqs (36) can be solved by using matrix inversion once θ and β are specified.

Example: Transform the Laplace equation of steady state fluid flow into the triangular coordinate system using θ= 60° & β=120°.

Solution: we know that the 2- dimensional form of Laplace equation is $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$;

using eq (36), we have

$$\begin{Bmatrix} \frac{\partial^2 T}{\partial q^2} \\ \frac{\partial^2 T}{\partial r^2} \\ \frac{\partial^2 T}{\partial s^2} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & \sqrt{3}/2 & 3/4 \\ 1/4 & -\sqrt{3}/2 & 3/4 \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 T}{\partial x^2} \\ \frac{\partial^2 T}{\partial x \partial y} \\ \frac{\partial^2 T}{\partial y^2} \end{Bmatrix}$$

Note that $\frac{\partial^2 T}{\partial x \partial y}$ is not needed in this case (i.e. since Laplace eq $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$).

$$\frac{\partial^2 T}{\partial q^2} = \frac{\partial^2 T}{\partial x^2} \quad \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow \frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial q^2}$$

$$\frac{\partial^2 T}{\partial r^2} = \frac{1}{4}\frac{\partial^2 T}{\partial x^2} + \frac{\sqrt{3}}{2}\frac{\partial^2 T}{\partial x \partial y} + 3/4\frac{\partial^2 T}{\partial y^2} \rightarrow\rightarrow \frac{\partial^2 T}{\partial x \partial y} = \frac{2}{\sqrt{3}}\left(\frac{\partial^2 T}{\partial r^2} - \frac{1}{4}\frac{\partial^2 T}{\partial x^2} - 3/4\frac{\partial^2 T}{\partial y^2}\right)$$

$$\frac{\partial^2 T}{\partial s^2} = \frac{1}{4}\frac{\partial^2 T}{\partial x^2} - \frac{\sqrt{3}}{2}\frac{\partial^2 T}{\partial x \partial y} + 3/4\frac{\partial^2 T}{\partial y^2} \rightarrow\rightarrow \frac{\partial^2 T}{\partial y^2} = 4/3\left(\frac{\partial^2 T}{\partial s^2} - \frac{1}{4}\frac{\partial^2 T}{\partial x^2} + \frac{\sqrt{3}}{2}\frac{\partial^2 T}{\partial x \partial y}\right)$$

$$\rightarrow\rightarrow \frac{\partial^2 T}{\partial y^2} = 4/3\left(\frac{\partial^2 T}{\partial s^2} - \frac{1}{4}\frac{\partial^2 T}{\partial q^2} + \frac{\sqrt{3}}{2}\left(\frac{2}{\sqrt{3}}\left(\frac{\partial^2 T}{\partial r^2} - \frac{1}{4}\frac{\partial^2 T}{\partial q^2} - 3/4\frac{\partial^2 T}{\partial y^2}\right)\right)\right)$$

$$\rightarrow\rightarrow \frac{\partial^2 T}{\partial y^2} = \left(\frac{4}{3}\frac{\partial^2 T}{\partial s^2} - \frac{1}{3}\frac{\partial^2 T}{\partial q^2} + \frac{4}{3}\frac{\partial^2 T}{\partial r^2} - \frac{1}{3}\frac{\partial^2 T}{\partial q^2} - \frac{\partial^2 T}{\partial y^2}\right)$$

$$\rightarrow\rightarrow \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial y^2} = \left(\frac{4}{3}\frac{\partial^2 T}{\partial s^2} - \frac{2}{3}\frac{\partial^2 T}{\partial q^2} + \frac{4}{3}\frac{\partial^2 T}{\partial r^2}\right) \rightarrow\rightarrow 2\frac{\partial^2 T}{\partial y^2} = \left(\frac{4}{3}\frac{\partial^2 T}{\partial s^2} - \frac{2}{3}\frac{\partial^2 T}{\partial q^2} + \frac{4}{3}\frac{\partial^2 T}{\partial r^2}\right)$$

$$\rightarrow\rightarrow \frac{\partial^2 T}{\partial y^2} = 1/2\left(\frac{4}{3}\frac{\partial^2 T}{\partial s^2} - \frac{2}{3}\frac{\partial^2 T}{\partial q^2} + \frac{4}{3}\frac{\partial^2 T}{\partial r^2}\right) \rightarrow\rightarrow \frac{\partial^2 T}{\partial y^2} = \frac{2}{3}\frac{\partial^2 T}{\partial s^2} - \frac{1}{3}\frac{\partial^2 T}{\partial q^2} + \frac{2}{3}\frac{\partial^2 T}{\partial r^2}$$

Substituting in Laplace eq. giving

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 T}{\partial q^2} + \frac{2}{3}\frac{\partial^2 T}{\partial s^2} - \frac{1}{3}\frac{\partial^2 T}{\partial q^2} + \frac{2}{3}\frac{\partial^2 T}{\partial r^2} = \frac{2}{3}\left(\frac{\partial^2 T}{\partial q^2} + \frac{\partial^2 T}{\partial r^2} + \frac{\partial^2 T}{\partial s^2}\right)$$

which is wanted.

Example: Assume that the steady state ***Laplace Equation;***

Find the steady state flow at node **2** in the attached **(Fig. 3)**

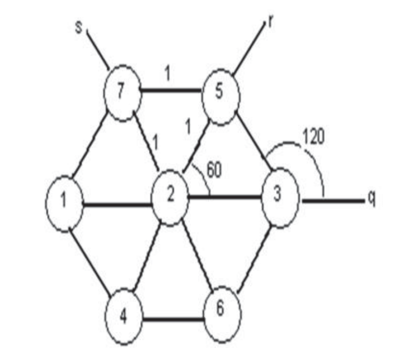


Fig. 3

Solution:

At node 2, the steady state is;

$$\frac{\partial^2 T}{\partial q^2} = \frac{1}{(\Delta q)^2}(T_1 - 2T_2 + T_3); \quad \frac{\partial^2 T}{\partial r^2} = \frac{1}{(\Delta r)^2}(T_4 - 2T_2 + T_5)$$

$$\frac{\partial^2 T}{\partial s^2} = \frac{1}{(\Delta s)^2}(T_6 - 2T_2 + T_7)$$

Since

$\Delta q = \Delta r = \Delta s = 1$

Hence Laplace steady state eq. is;

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{2}{3}\left(\frac{\partial^2 T}{\partial q^2} + \frac{\partial^2 T}{\partial r^2} + \frac{\partial^2 T}{\partial s^2}\right)$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{2}{3*1}(T_1 - 2T_2 + T_3 + T_4 - 2T_2 + T_5 + T_6 - 2T_2 + T_7)$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{2}{3}(T_1 - 6T_2 + T_3 + T_4 + T_5 + T_6 + T_7)$$