# Chapter 11.04
# Discrete Fourier Transform

## Introduction

Recalled the exponential form of Fourier series (see Equations 18 and 20 from Chapter 11.02),

$$f(t) = \sum_{k=-\infty}^{\infty} \widetilde{C}_k e^{ikw_0 t} \qquad (18, \text{Ch. } 11.02)$$

$$\widetilde{C}_k = \left(\frac{1}{T}\right)\left\{\int_0^T f(t) \times e^{-ikw_0 t} dt\right\} \qquad (20, \text{Ch. } 11.02)$$

While the above integral can be used to compute $\widetilde{C}_k$, it is more preferable to have a discretized formula version to compute $\widetilde{C}_k$. Furthermore, the Discrete Fourier Transform (or DFT) [1–5] will also facilitate the development of much more efficient algorithms for Fast Fourier Transform (or FFT), to be discussed in Chapters 11.05 and 11.06.

## Derivations of DFT Formulas

If time "$t$" is discretized at $t_1 = \Delta t, t_2 = 2\Delta t, t_3 = 3\Delta t, \ldots\ldots., t_n = n\Delta t,$

Then Equation (18, of Chapter 11.02) becomes

$$f(t_n) = \sum_{k=0}^{N-1} \widetilde{C}_k e^{ikw_0 t_n} \qquad (1)$$

To simplify the notation, define

$$t_n = n \qquad (2)$$

Then, Equations (1) can be written as

$$f(n) = \sum_{k=0}^{N-1} \widetilde{C}_k e^{ikw_0 n} \qquad (3)$$

In the above formula, "$n$" is an integer counter. However, $f(n)$ and $t_n$ do NOT have to be integer numbers.

Multiplying both sides of Equation (3) by $e^{-ilw_0 n}$, and performing the summation on "$n$", one obtains ( note: $l$ = integer number)

$$\sum_{n=0}^{N-1} f(n) \times e^{-ilw_0 n} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} \tilde{C}_k e^{ikw_0 n} \times e^{-ilw_0 n} \tag{4}$$

$$= \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} \tilde{C}_k e^{i(k-l)w_0 n} \tag{5}$$

$$= \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} \tilde{C}_k e^{i(k-l)\frac{2\pi}{N}n} \tag{6}$$

Switching the order of summations on the right-hand-side of Equation (6), one obtains

$$\sum_{n=0}^{N-1} f(n) \times e^{-il\left(\frac{2\pi}{N}\right)n} = \sum_{k=0}^{N-1} \tilde{C}_k \sum_{n=0}^{N-1} e^{i(k-l)\left(\frac{2\pi}{N}\right)n} \tag{7}$$

Define

$$A = \sum_{n=0}^{N-1} e^{i(k-l)\left(\frac{2\pi}{N}\right)n} \tag{8}$$

There are 2 possibilities for $(k-1)$ to be considered in Equation (8)

**Case(1):** $(k - l)$ is a multiple integer of $N$, such as

$(k - l) = mN$; or $k = l + mN$ where $m = 0, \pm 1, \pm 2, ......$

Thus, Equation (8) becomes:

$$A = \sum_{n=0}^{N-1} e^{im2\pi n} \tag{9}$$

$$= \sum_{n=0}^{N-1} \cos(mn2\pi) + i\sin(mn2\pi)$$

Hence:

$$A = N \tag{10}$$

**Case(2):** $(k - l)$ is NOT a multiple integer of $N$

In this case, from Equation (8) one has

$$A = \sum_{n=0}^{N-1} \left\{ e^{i(k-l)\left(\frac{2\pi}{N}\right)} \right\}^n \tag{11}$$

Define:

$$a = e^{i(k-l)\frac{2\pi}{N}} \tag{12}$$

$$= \cos\left\{(k-l)\frac{2\pi}{N})\right\} + i\sin\left\{(k-l)\frac{2\pi}{N})\right\}$$

$a \neq 1$; because $(k - l)$ is "NOT" a multiple integer of $N$ (13)

Then, Equation (11) can be expressed as

$$A = \sum_{n=0}^{N-1} a^n \tag{14}$$

From mathematical handbooks, the right side of Equation (14) represents the "geometric series", and can be expressed as

$$A = \sum_{n=0}^{N-1} a^n = N \text{ if } a = 1 \tag{15}$$

$$= \frac{1-a^N}{1-a} \text{ if } a \neq 1 \tag{16}$$

Because of Equation (13), hence Equation (16) should be used to compute $A$. Thus

$$A = \frac{1-a^N}{1-a} \text{ (See Equation (12))} \tag{17}$$

$$= \frac{1-e^{i(k-l)2\pi}}{1-a}$$

Since ($k-1$) is still a multiple of $2\pi$, hence

$$e^{i(k-l)2\pi} \equiv \cos\{(k-l)2\pi\} + i\sin\{(k-l)2\pi\} \tag{18}$$

$$= 1$$

Substituting Equation (17) into Equation (18), one gets

$$A = 0 \tag{19}$$

Thus, combining the results of case (1) and case (2), one gets (see Equations (10) and Equation (19))

$$A = N + 0 \tag{20}$$

Substituting Equation (20) into Equation (8), and then referring to Equation (7), one gets

$$\sum_{n=0}^{N-1} f(n) e^{-ilw_0 n} = \sum_{k=0}^{N-1} \tilde{C}_k \times N \tag{20a}$$

Recalled $k = l + mN$ (where $l, m$ are integer numbers), and since $k$ must be in the range $0 \to N-1$, therefore $m = 0$. Thus:

$$k = l + mN \text{ becomes } k = l$$

Equation (20a) can, therefore, be simplified to

$$\sum_{n=0}^{N-1} f(n) e^{-ilw_0 n} = \tilde{C}_l \times N \tag{20b}$$

Thus

$$\tilde{C}_l = \tilde{C}_k = \left(\frac{1}{N}\right) \sum_{n=0}^{N-1} f(n) e^{-ikw_0 n} \tag{21}$$

$$= \left(\frac{1}{N}\right) \sum_{n=0}^{N-1} f(n) \{\cos(kw_0 n) - i\sin(kw_0 n)\}$$

where

$$n \equiv t_n$$

and

$$f(n) = \sum_{k=0}^{N-1} \tilde{C}_k \, e^{ikw_0 n} \tag{3, repeated}$$

$$= \sum_{k=0}^{N-1} \tilde{C}_k \{\cos(kw_0 n) + i\sin(kw_0 n)\}$$

Remarks:

(a) Consider the exponential term in Equation (1). Let

$$E = e^{(ikw_0 n)} = e^{(ik\times\frac{2\pi}{N}*n)}$$

If one replaces "$n$" by "$-(N-n)$" (or "$(n-N)$") into the above equation, then one obtains

$$e^{ik\times\frac{2\pi}{N}\times(n-N)} = e^{(ik\times\frac{2\pi}{N}*n)}\times\left[e^{(-ik\times 2\pi)}=1\right]$$
$$= E$$

Thus, Equation (1) indicates that the force corresponding to frequencies of order "$n$" and "$-(N-n)=n-N$" have the same values. Hence

$$w_n = n\overline{w} \quad \text{for } n \le \frac{N}{2}$$
$$= -(N-n)\overline{w} \quad \text{for } n > \frac{N}{2}$$

and the frequency corresponding to $n=\frac{N}{2}$ is the highest frequency that can be considered in the discrete Fourier series ($w_{\frac{N}{2}}$ is called the Nyquist frequency). If there are harmonic (force) components above $w_{\frac{N}{2}}$ in the original function, then these higher components will introduce distortions in the lower harmonic components (known as ALIASING phenomenon). Because of the ALIASING phenomenon, the number of ($N$) data points should be "at least twice" the highest harmonic component presents in the (forcing) function, for sufficient computational accuracy. As an example, if the forcing function is given as

$$F(t)=\sum_{n=1}^{16} 100\times\cos(2\pi nt)$$

then, the minimum value of $N$ ( = Number of sample data points ) should be $N_{\min} = 32$.

(b) The factor $\left(\frac{1}{N}\right)$, shown in the DFT Equation (21), is merely a scale factor. It can also be placed in the inverse Fourier Transform Equation (1), but not both.

Thus, Equations (21) and (1) can be re written as

$$\tilde{C}_n = \sum_{k=0}^{N-1} f(k)e^{-ik\left(w_0=\frac{2\pi}{N}\right)n} \tag{22}$$

$$f(k)=\left(\frac{1}{N}\right)\sum_{n=0}^{N-1}\tilde{C}_n e^{ik\left(w_0=\frac{2\pi}{N}\right)n} \tag{23}$$

To avoid computation with "complex numbers", Equation (22) can be expressed as

$$\tilde{C}_n^R + i\tilde{C}_n^I = \sum_{k=0}^{N-1}\left\{f^R(k)+i\quad f^I(k)\right\}\times\left\{\cos(\theta)-i\sin(\theta)\right\} \tag{22a}$$

where

$$\theta = k\left(w_0=\frac{2\pi}{N}\right)n \tag{22b}$$

$$\tilde{C}_n^R + i\tilde{C}_n^I = \sum_{k=0}^{N-1}\left\{f^R(k)\times\cos(\theta)+f^I(k)\sin(\theta)\right\}+i\left\{f^I(k)\cos(\theta)-f^R(k)\sin(\theta)\right\}$$

The above "complex number" equation is equivalent to the following 2 "real number" equations

$$\tilde{C}_n^R = \sum_{k=0}^{N-1}\left\{f^R(k)\cos(\theta) + f^I(k)\sin(\theta)\right\} \tag{22c}$$

$$\tilde{C}_n^I = \sum_{k=0}^{N-1}\left\{f^I(k)\cos(\theta) - f^R(k)\sin(\theta)\right\} \tag{22d}$$

Computer program implementation for the DFT equations (22c, 22d) are given at http://numericalmethods.eng.usf.edu/simulations/mtl/11fft/dft.m .

**Detailed Explanation About Aliasing Phenomenon, Nyquist Samples, Nyquist Rate.**

When a function $f(t)$, which may represent the signals from some real-life phenomenon (shown in Figure 1), is sampled, it basically converts that function into a sequence $\tilde{f}(k)$ at discrete locations of $t$. These discrete locations are assumed to have "equally spaced and the distance between any 2 samples is $\Delta t$. Thus, $\tilde{f}(k)$ represents the value of $f(t)$, at $t = t_0 + k\Delta t$, where $t_0$ is the location of the first sample $(at\ k = 0)$. If the sample locations were done properly, then the original function $f(t)$, can be recovered through interpolation process of these discrete sample values.

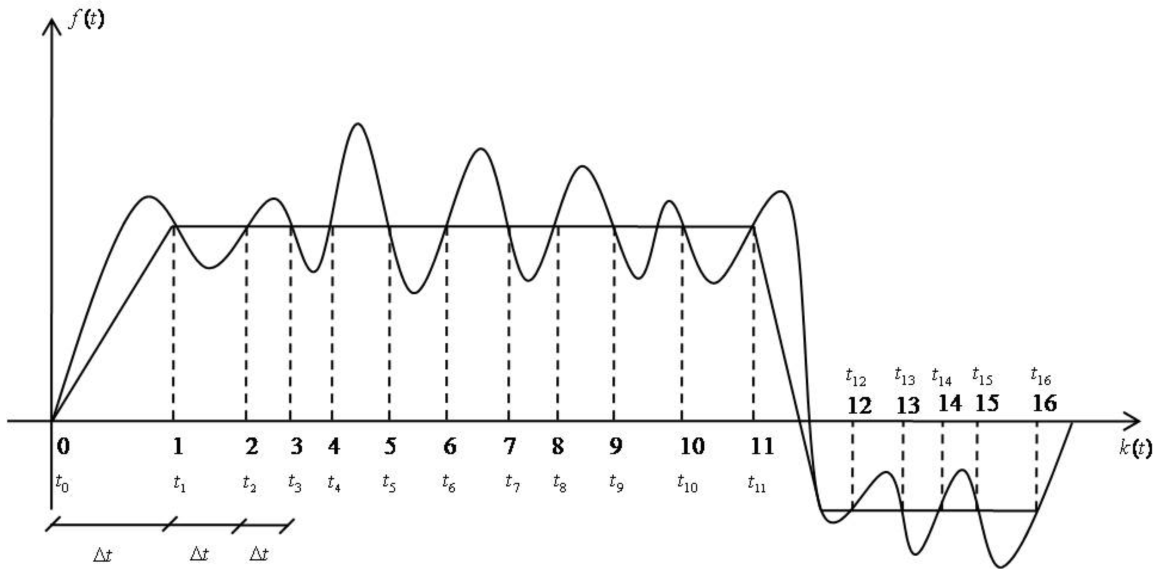


**Figure 1** Function to be Sampled and "Aliased" Sample Problem.

In Figure 1, the samples have been taken with a fairly large $\Delta t$. Thus, these sequence of discrete data will not be able to recover the original signal function $f(t)$. For example, if all discrete values of $f(t)$, were connected by piecewise linear fashion, then a nearly horizontal straight line will occur between $t_1$ through $t_{11}$, and $t_{12}$ through $t_{16}$, respectively (See Figure 1). These piecewise linear interpolation (or other interpolation schemes will NOT produce a curve which resemble well with the original function $f(t)$. This is the case where the data has been "ALIASED".

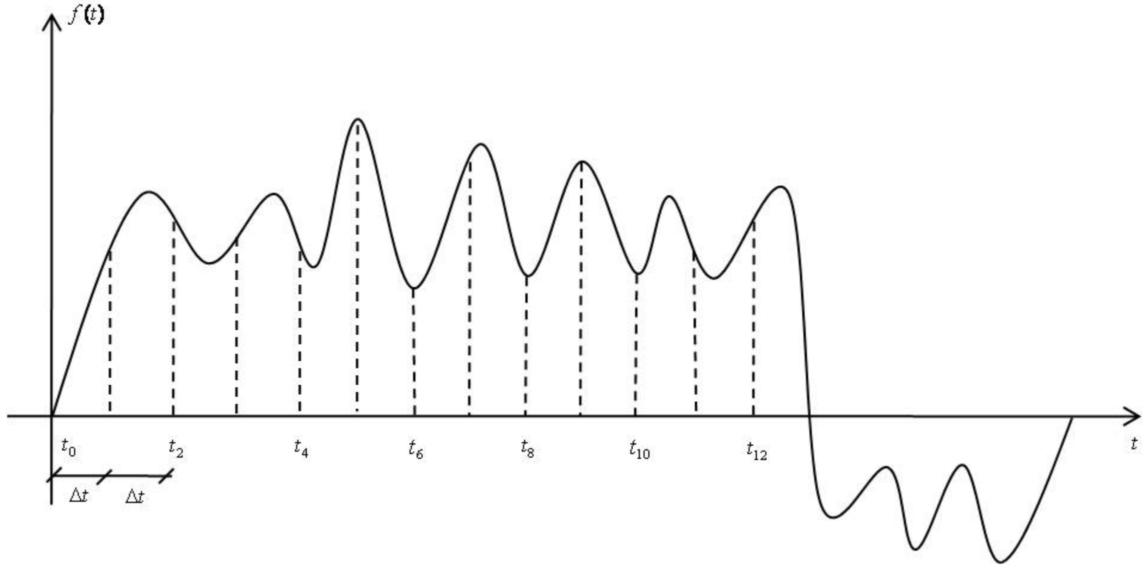


**Figure 2** Function to be sampled and "Windowing" Sample Problem.

Another potential difficulty in sampling the function is called "windowing" problem. As indicated in Figure 2, while $\Delta t$ is small enough so that a piecewise linear interpolation for connecting these discrete values will adequately resemble the original function $f(t)$, however, only a portion of the function $f(t)$ has been sampled (from $t_1$ through $t_{12}$) rather than the entire one. In other words, one has placed a "window" over the function.

To avoid aliased phenomenon, the sample space $\Delta t$ should be small enough so that the discrete sequence will recover back the original function $f(t)$. The "sampling theorem" can be stated as:

"If the function $f(t)$ is band-limited with bandwidth $2w_{\max}$, $F(w) \equiv$ Fourier transform of $f(t)=0$ for $|w| \geq w_{\max} > 0$ then $f(t)$ is uniquely determined by a knowledge of its values at uniformly spaced intervals $\Delta t$ apart, with $\Delta t = \dfrac{1}{2w_{\max}}$.

The above "sampling theorem" can be loosely explained through the help of Figure 3.

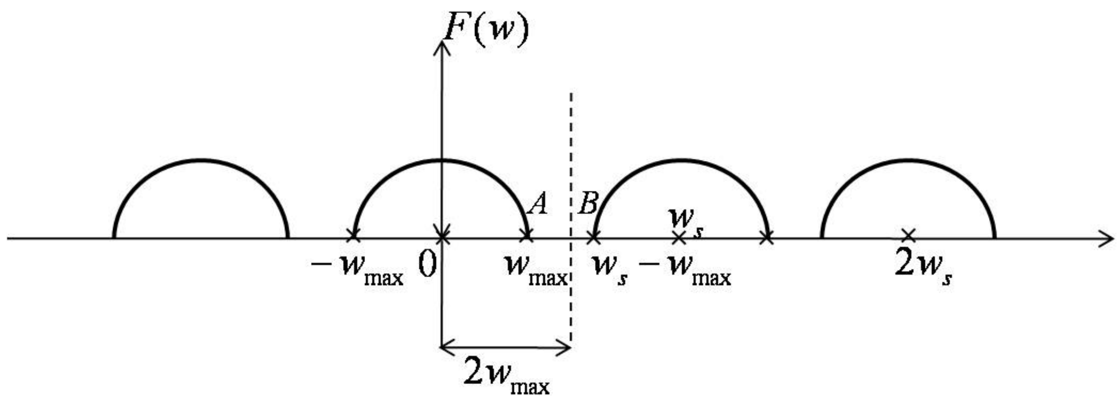


**Figure 3** Frequency of sampling rate ( $w_S$ ) versus maximum frequency content ( $w_{max}$ ).

To satisfy $F(w)=0$, for $|w| \geq w_{max}$, the frequency ( $w$ ) should be between points $A$ and $B$ of Figure 3.

Hence

$$w_{max} \leq w \leq w_s - w_{max}$$

which implies

$$w_s \geq 2w_{max}$$

Physically, the above equation states that one must have at least 2 samples per cycle of the highest frequency component present (Nyquist samples, Nyquist rate).

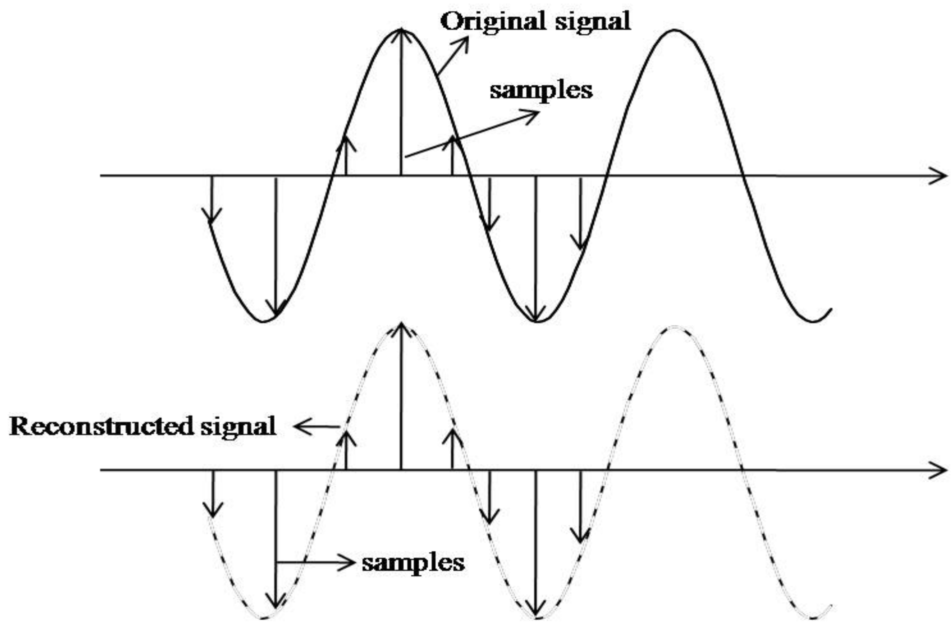


**Figure 4** Correctly reconstructed signal.

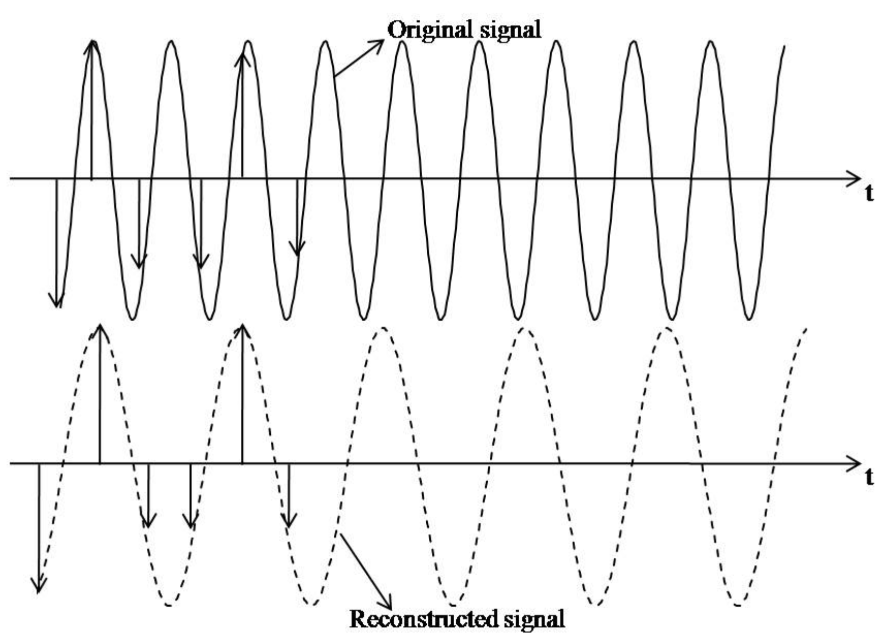


**Figure 5** Wrongly reconstructed signal.

In Figure 4, a sinusoidal signal is sampled at the rate of 6 samples per 1 cycle (or $w_s = 6w_0$). Since this sampling rate does satisfy the sampling theorem requirement $\left(w_s \geq 2w_{\max}\right)$, the reconstructed signal does correctly represent the original signal. However, as indicated in Figure 5 a sinusoidal signal is sampled at the rate of 6 samples per 4 cycles $\left(or\ w_s = \frac{6}{4} w_0\right)$. Since this sampling rate does NOT satisfy the requirement $\left(w_s \geq 2w_{\max}\right)$, the reconstructed signal would wrongly represent the original signal.

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