Capacitance finding by Laplace s equation

Examples of the solution of Laplace s equation

Several methods have been developed for solving Laplace’s equation. The simplest method is that of direct integration. We will use this technique to work several examples involving one – dimensional potential variation in various coordinate systems in this section.

هناك عدة طرق لحل معادلة لابلاس , الطريقة الاسهل هي بالتكامل المباشر. لذا سوف نستخدم هذه الطريقة لنطبقها في عدد من الامثلة مقتصرين على حالة (تغير الجهد مع بعد واحد فقط) وذلك في مختلف انظمة الاحداثيات. اليك طريقة الحل:

1) Given V , use $\mathbf{E} = -\nabla V$ to find **E**.
2) Use $\mathbf{D} = \epsilon\mathbf{E}$ to find **D**.
3) Evaluate **D** at either capacitor plate, $\mathbf{D} = \mathbf{D_s} = \mathbf{D}_N \mathbf{a}_N$.
4) recognize that $\rho s = D_N$.
5) Find Q by a surface integration over the capacitor plate, $Q = \int_s \rho s \, ds$.

- In rectangular coordinate, let us assume that **V is a function of x only**:

Laplace’s equation reduced to $$\frac{\partial^2 V}{\partial x^2} = 0$$

Since V is not a function of y or z then: $$\frac{d^2 V}{dx^2} = 0$$

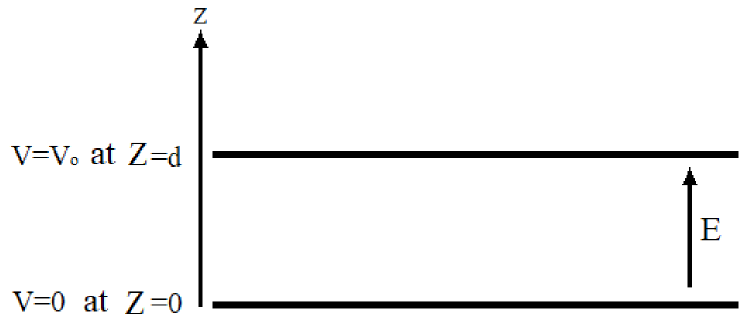
By integration: $$\frac{dv}{dx} = A$$

Another integration: $$V = Ax + B$$

where A and B are constants of integration

Example : Find the capacitance of a parallel-plate area *S*, plate separation *d*, and potential difference V_o between plates, (*let the potential variation is with Z only*)?

Solution:



$$\frac{\partial^2 V}{\partial z^2} = 0$$

V = Az + B …….(1)

To find the values of A and B:

V=0 at Z=0

0=A(0) +B $\Longrightarrow$ B=0

Eq (1) became: V =AZ ……..(2)

V=V∘ at Z=d $\Longrightarrow$ V∘ = Ad $\Longrightarrow$ $A = \frac{V_\circ}{d}$

Subst. A in (2)

$$V = \frac{V_\circ}{d} Z$$

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial z} az = -\frac{V_o}{d} az$$

$$\mathbf{D} = \epsilon \mathbf{E} = -\frac{\epsilon V_o}{d} az$$

$$\rho s = D_N = -\frac{\epsilon V_o}{d}$$

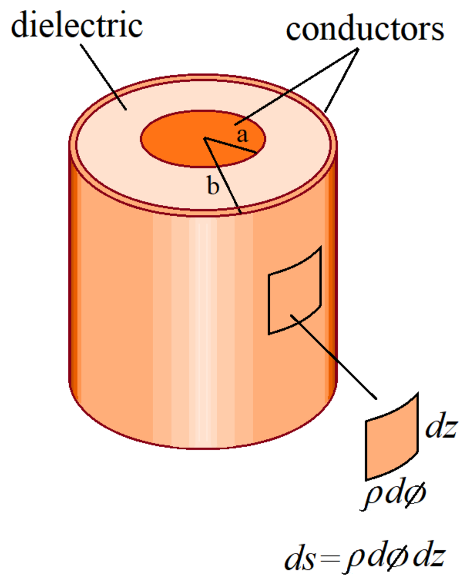
$$Q = \int_s \rho s \, ds = \int_s -\frac{\epsilon V_o}{d} ds = -\frac{\epsilon V_o S}{d}$$

$$C = \frac{Q}{V_o} = \frac{\epsilon S}{d}$$

- In cylindrical coordinate, let us assume that **V is a function of ρ only**:

Example : Find the capacitance of coaxial transmission line?

Solution: between the two conductors of coaxial transmission line, V is a function of ρ only:



$$\nabla^2 V = 0 \Longrightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = 0$$

Noting the ρ in the denominator,

we exclude ρ=0 from our solution

and then multiply by ρ and integrate:

$$\left(\rho\frac{dV}{d\rho}\right) = \text{A}$$

Where a total derivative replaces the partial derivative because V varies only with ρ. Next, rearrange, and integrate again:

$V = A\, ln\rho + B$ …….(1)

$V=0$ at $\rho = b$

$0= A\, lnb + B \Longrightarrow B= -A\, lnb$

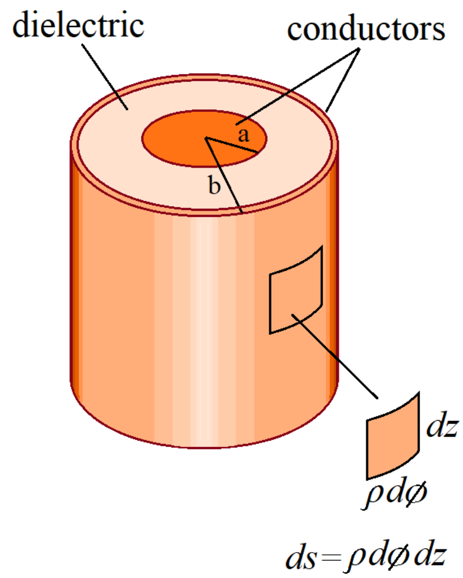
Eq (1) became: $V = A\, ln\rho - A\, lnb$

$V=A\, ln\frac{\rho}{b}$ ……..(2)

$V=V_o$ at $\rho=a \Longrightarrow V_o = A\, ln\frac{a}{b} \Longrightarrow A = \frac{V_o}{ln\frac{a}{b}}$

Subst. A in (2)

$$V = \frac{V_\circ \, ln\, \frac{\rho}{b}}{ln\, \frac{a}{b}}$$



$$\mathbf{E} = -\nabla V$$

$$\mathbf{E} = \frac{V_\circ}{\rho ln\frac{b}{a}}\, a_\rho$$

$$\mathbf{D} = \epsilon\mathbf{E} = \frac{\epsilon V_\circ}{\rho ln\frac{b}{a}}\, a_\rho = \mathbf{D_N}$$

$$\rho s = D_N = \frac{\epsilon V_\circ}{\rho ln\frac{b}{a}} \qquad \text{(scalar)}$$

$$Q = \int_s \rho s \, ds = \int_{z=0}^{L} \int_{\emptyset=0}^{2\pi} \frac{\epsilon V_\circ}{\rho ln\frac{b}{a}}\, \rho d\emptyset\, dz$$

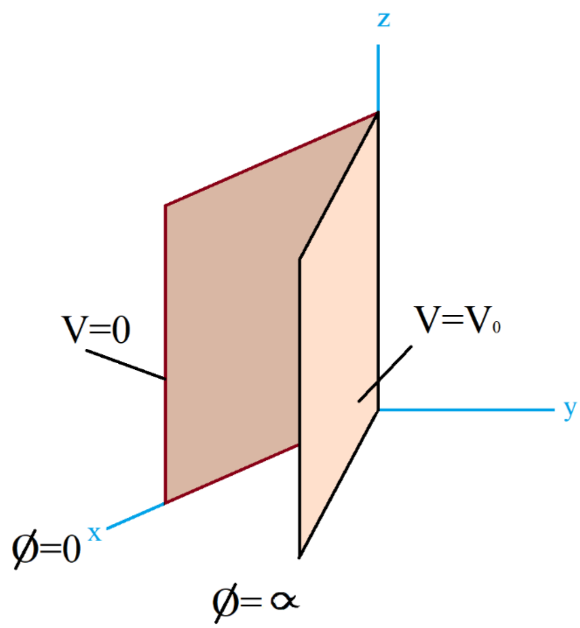
$$Q = \frac{2\pi L\epsilon V_\circ}{ln\frac{b}{a}}$$

$$C = \frac{Q}{V_\circ} = \frac{2\pi L\epsilon}{ln\,\frac{b}{a}}$$

- In cylindrical coordinate, let us assume that **V is a function of *Φ* only**:
  Laplace's equation :
  $$\nabla^2 V = 0$$
  $$\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \emptyset^2} = 0$$

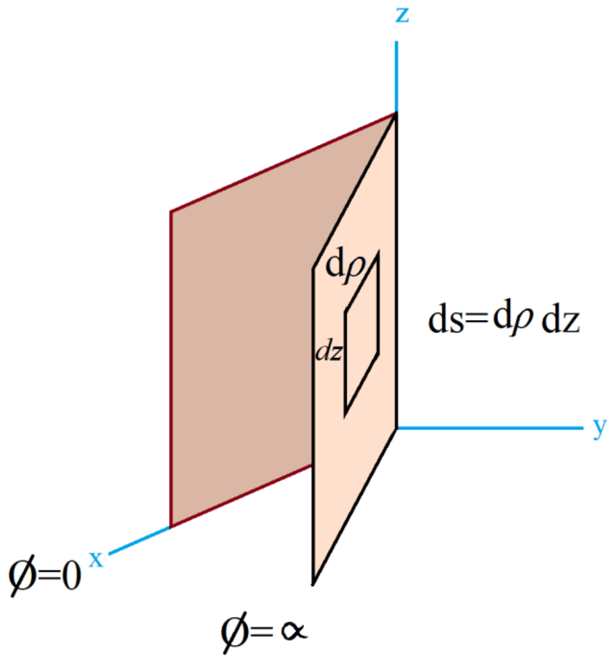
  we exclude ρ=0 from our solution and multiply by ρ^2
  $$\frac{\partial^2 V}{\partial \emptyset^2} = 0$$
  and then integrate:
  $$\left(\frac{dV}{d\emptyset}\right) = A$$
  Another integration
  $$V = A\,\emptyset + B \;.......(1)$$
  V=0 at $\emptyset = 0$
  $$0 = A\,(0) + B$$
  $$B = 0$$
  Eq (1) became:
  $$V = A\,\emptyset ....(2)$$
  $V = V_o$ at $\emptyset = \alpha \Longrightarrow V_o = A\alpha$
  $$A = \frac{V_\circ}{\alpha}$$
  Subst. A in (2)
  $$V = \frac{V_\circ}{\alpha}\emptyset$$

  $$\mathbf{E} = -\nabla V$$
  $$\mathbf{E} = -\frac{1}{\rho}\frac{\partial V}{\partial \emptyset}\, a_\emptyset$$
  $$\mathbf{E} = -\frac{1}{\rho}\frac{\partial}{\partial \emptyset}\left(\frac{V_\circ}{\alpha}\emptyset\right)\, a_\emptyset$$
  $$\mathbf{E} = -\frac{1}{\rho}\left(\frac{V_\circ}{\alpha}\right)\, a_\emptyset$$



$\mathbf{D} = \epsilon\mathbf{E}$

$\mathbf{D} = -\frac{V_\circ \epsilon}{\rho \alpha}\ a_\emptyset$



$\rho s = D_N$

$\rho s = -\frac{V_\circ \epsilon}{\rho \alpha}$ *(scalar)*

$$Q = \int_s \rho s\ ds$$

Assume minimum ρ=x (the isolator thickness)

$$Q = \int_{z=0}^{L} \int_{\rho \simeq x}^{\rho} \frac{-\epsilon V_\circ}{\rho \alpha}\ d\rho\ dz$$

$Q = \frac{V_\circ \epsilon}{\alpha}\ L\ ln \frac{\rho}{x}$

$$C = \frac{Q}{V_\mathrm{o}} = \frac{\epsilon}{\alpha}\ L\ ln \frac{\rho}{x}$$