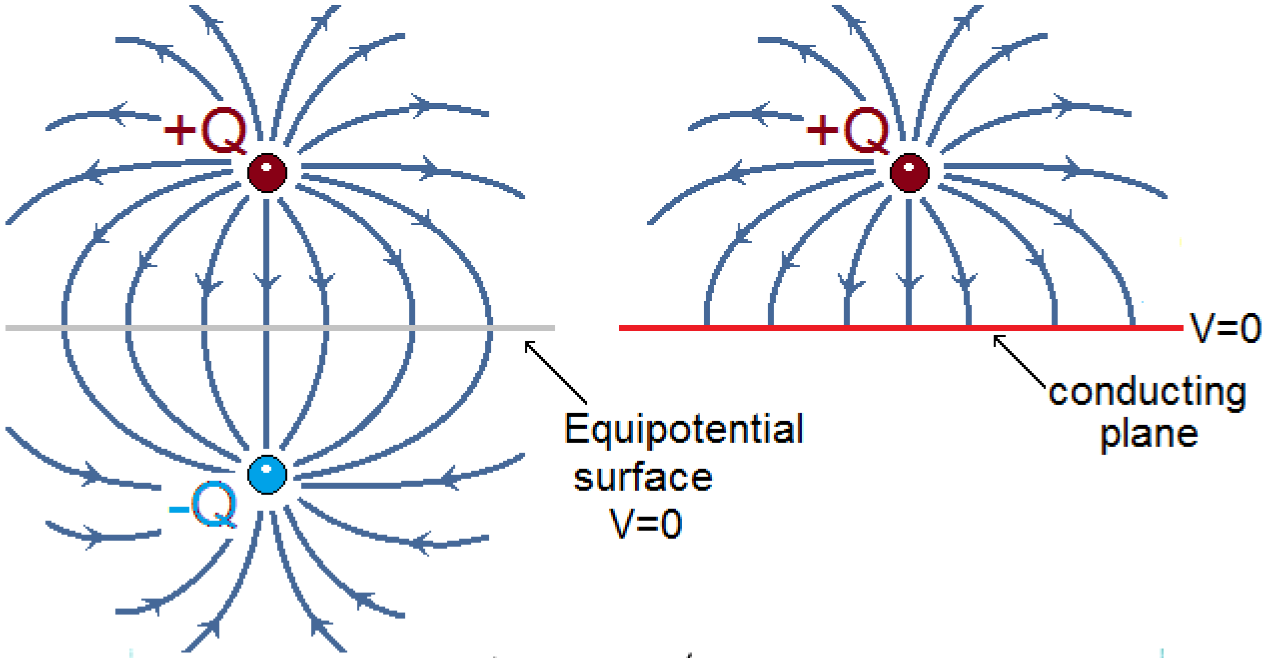
The Method of Images P: 124

For a dipole field:
We can replace the infinite zero potential plane by a vanishingly thin infinite conducting plane with zero potential V = 0. and the electric field is therefor normal to the surface. and by a single charge only, the field will remain the same in the single charge side. in the other side all fields are zero as we have not provided any charges in that region.



also the opposite is right, if a single charge above a perfectly conducting plane, we may maintain the same field above the plane by removing this conducting plane and locating a negative charge at a symmetrical location below the plane. This charge is called the image of the original charge.
This is the same for other charge configurations.
In many cases the potential field of the new system is much easier to find.

Example: Let us find the surface charge density at P(2,5,0) on the conducting plane Z=0 if there is a line charge of 30 nc/m located at X=0 , Z=3.

Solution: The line charge and the conducting plane are shown in figure 1 below.

The conducting plane is will act as a mirror, and then an image of the line charge will be created in the other side of the plane but with an opposite signe ie ρ_L(image) = -30 nc/m as shown in figure 2.

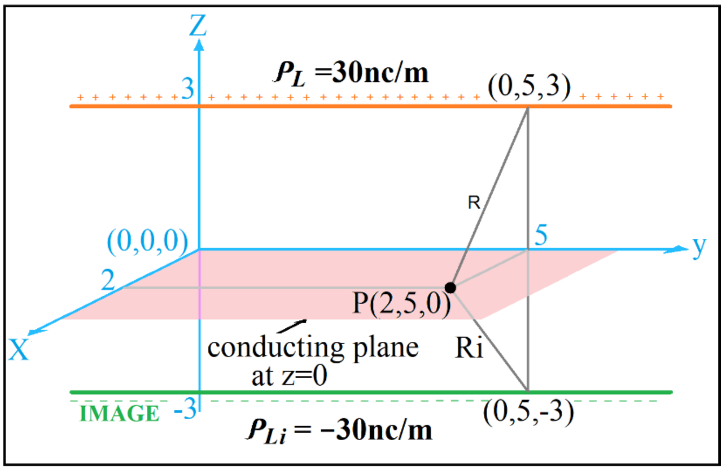


Figure (2)

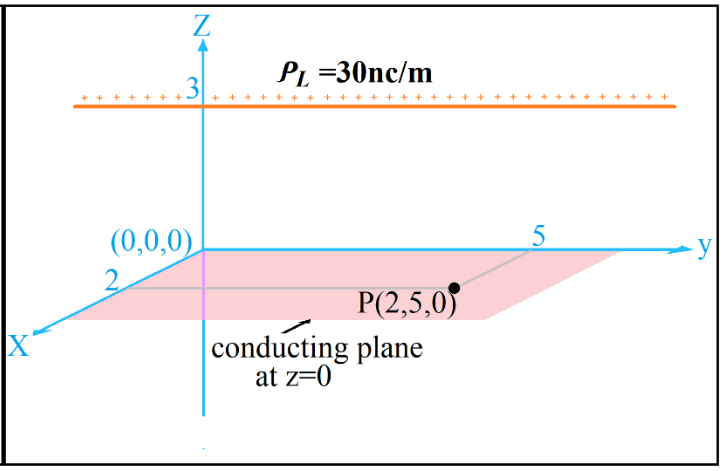


Figure (1)

$$\bar{E}_+ = \frac{\rho_L}{2\pi\epsilon_0 R_+} a_{R+} = \frac{30\times10^{-9}}{2\pi\epsilon_0\sqrt{13}} \cdot \frac{(2-0)\bar{a}_x + (0-3)\bar{a}_z}{\sqrt{13}}$$

$$\bar{E}_- = \frac{-30\times10^{-9}}{2\pi\epsilon_0\sqrt{13}} \cdot \frac{(2-0)a_x + (0-(-3))\bar{a}_z}{\sqrt{13}} = \frac{30\times10^{-9}}{2\pi\epsilon_0\sqrt{13}} \cdot \frac{-2\bar{a}_x - 3\bar{a}_z}{\sqrt{13}}$$

$$\bar{E}_T = \bar{E}_+ + \bar{E}_- = \frac{-180\times10^{-9}\bar{a}_z}{2\pi\epsilon_0(13)} = -249\,\bar{a}_z \ \ V/m$$

∴ the field is normal to the conducting plain.

$$\bar{D} = \epsilon_0\bar{E} = 8.854\times10^{-12} \times (-249)\bar{a}_z = -2.2\,\bar{a}_z \ \ nC/m^2$$

$$\therefore \rho_s\Big|_{at\ P} = -2.2\ nC/m^2.$$

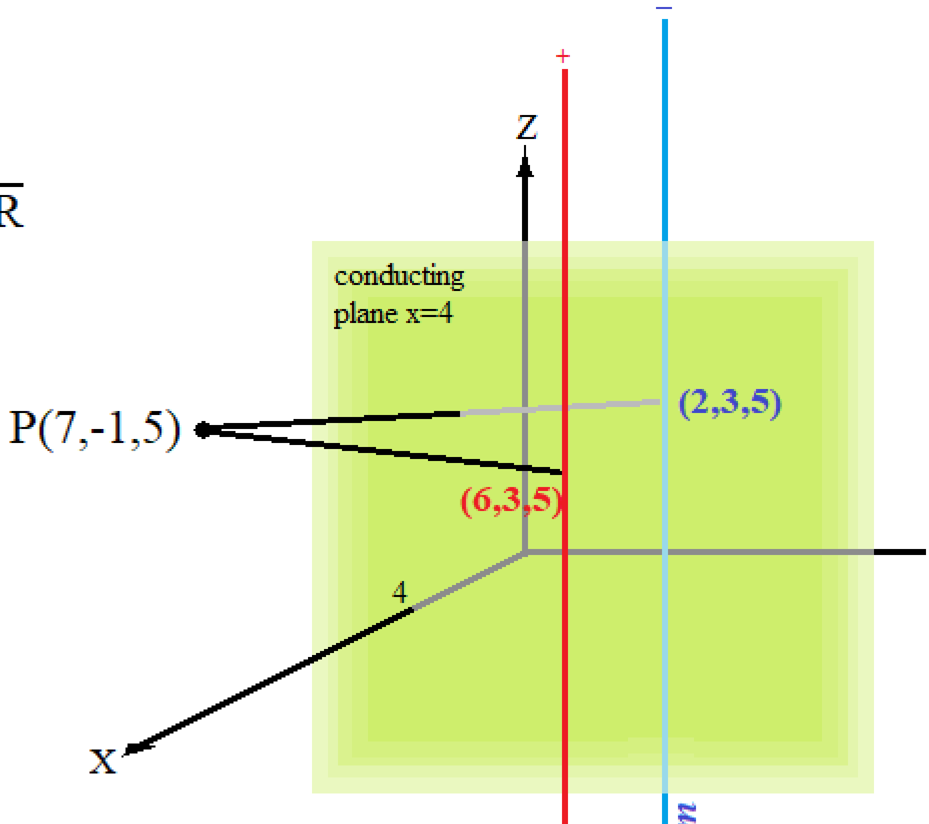
D 5.6

A perfect conducting plane is located in free space at X=4 , and a uniform infinite line charge of 40 nc/m lies along the line X=6, y=3 let V=0 at the conducting plane. At P(7,-1,5) find: a) V , b) **E**

Solution:

V at P(7,-1,5):

$$E+\Big|_{p} = \frac{\rho_L}{2\pi\epsilon_\circ R}\,\overline{a_R}$$



$$= \frac{40\times10^{-9}}{2\pi\epsilon_\circ\sqrt{1^2+4^2}}\;\frac{(7-6)\overline{a_x}+[(-1)-3]\overline{a_y}}{\sqrt{1^2+4^2}}$$

$$E_+ = \frac{40\times10^{-9}}{2\pi\epsilon_\circ\sqrt{17}}\;\frac{\overline{a_x}-4\overline{a_y}}{\sqrt{17}} = \frac{40\times10^{-9}}{2\pi\epsilon_\circ * 17}\cdot(a_x-4a_y)$$

$\rho_L=40$ nc/m
x=6 , y=3

$$E_- = \frac{-40\times10^{-9}}{2\pi\epsilon_\circ\sqrt{5^2+4^2}}\;\frac{(7-2)\overline{a_x}+[(-1)-3]\overline{a_y}}{\sqrt{5^2+4^2}}$$

$\rho_L=-40$ nc/m
x=2 , y=3
(Image)

$$E_- = \frac{-40\times10^{-9}}{2\pi\epsilon_\circ\sqrt{41}}\;\frac{5\,\overline{a_x}-4\,\overline{a_y}}{\sqrt{41}} = \frac{-40\times10^{-9}}{2\pi\epsilon_\circ * 41}(5\overline{a_x}-4\overline{a_y})$$

$$\overline{E}_T = \overline{E}_+ + \overline{E}_- = \frac{40\times10^{-9}}{2\pi\epsilon_\circ}\left[\left(\frac{1}{17}-\frac{5}{41}\right)\overline{a_x}+\left(\frac{-4}{17}+\frac{4}{41}\right)\overline{a_y}\right]$$

$$\overline{E}_T = -45.4\,\overline{a_x} - 99.03\,\overline{a_y}$$

لايجاد الجهد في النقطة (7,-1,5) :

يجب حساب الجهد عندها من جراء الخط المشحون الاصلي + الجهد فيها من جراء الصورة

$V_{at\ p(7,-1,5)} = V\ (\text{line charge}) + V(\text{image})$

VAB= VA – VB

Let VB = 0 (for point B at ∞)

$$Vab = \frac{\rho_L}{2\pi\epsilon_\circ}\ln\frac{b}{a}$$

$$Vab = \frac{40\times10^{-9}}{2\pi\frac{10^{-9}}{36\pi}}\ (lnb - lna)$$

For the line charge $a = \sqrt{1^2+4^2} = \sqrt{17}$

For b at infinity Vb=0 , (b=∞)

$Vab_{(\text{Line charge})} = 720\ \ln b - 720\ \ln\sqrt{17}$

$Vab_{(\text{Image})} = -720\ \ln b + 720\ \ln\sqrt{41}$

$V_{at\ p(7,-1,5)} = V\ (\text{line charge}) + V(\text{image})$

$$Va = 720(ln\sqrt{41} - ln\sqrt{17}\ \ \cong 316.93\ (V0lt)$$

5.6 Semiconductors أشباه الموصلات

في اعلى حزمة التكافؤ ، فان الالكترونات هناك عندما تكتسب
طاقة كافية (غالباً ما تكون طاقة حرارية) فانها تعبر
المنطقة الخالية الصغيرة نسبياً (في أشباه الموصلات) لتصل
الى حزمة التوصيل conduction band تاركة خلفها فجوة
صغيرة تسمى hole (فراغ او فجوة) وعند انتقال الالكترون
من فجوة الى اخرى ، فهذا يعني ان الفجوة ايضاً ستنتقل
من ذرة الى اخرى ولكن باتجاه معاكس . سنعامل الفجوة
على انها شحنة موجبة ومساوية لشحنة الالكترون
وبانتقال كلا النوعين (الالكترونات والفجوات) يكون سريان
التيار .

$$\sigma = -\rho_e \mu_e + \rho_h \mu_h$$

At: 300k

mobility μ_{eh}	*electron* μ_e	*hole* μ_h
Pure Silicon	*0.12*	*0.025*
Pure Germanume	*0.36*	*0.17*

At: 300k

يعتمد تركيز الألكترونات والفجوات بشكل كبير على الحرارة:

$T\uparrow \quad \mu\downarrow \quad \rho_v\uparrow\uparrow \quad \sigma\uparrow$

يمكن ان نلاحظ ما يلي:

- **تقل مقاومة اشباه الموصلات مع زيادة الحرارة وبذلك فانها تتصرف عكس تصرف المعادن (الموصلات) وهذا من الفروق المهمة بين المعادن وأشباه الموصلات .**
- **باضافة الشوائب impurities ، فإن حاملات الشحنات تزداد وبالتالي تزداد التوصيلية .**

بالنسبة لأشباه الموصلات النقية ، فإن الصيغة النقطية لقانون كولوم تتحقق إذ أن التوصيلية ثابتة تقريبا مع تغير كثافة التيار أو إتجاهية التيار.

وعند عملية التشويب (doping) نحصل على نوعين من المواد:

- نحصل على مادة من نوع (N type) عند إضافة الشوائب الواهبة (Donor) التي تعطي الألكترونات.
- نحصل على مادة من نوع (P type) عند إضافة الشوائب المستقبلة (Ecceptors) التي تعطي الفجوات.

ملاحظة: بإضافة نسبة قليلة جدا من الشوائب في عملية (التشويب) يمكن مضاعفة التوصيلية بشكل كبير :

$$\frac{1}{10^7} \rightarrow \sigma * 10^5$$

D5.7. Using the values given in this section for the electron and hole mobilities in silicon at 300 K, and assuming hole and electron charge densities are 0.0029 C/m³ and -0.0029 C/m³, respectively, find: (*a*) the component of the conductivity due to holes; (*b*) the component of the conductivity due to electrons; (*c*) the conductivity.

Ans. 72.5 μS/m; 348 μS/m; 421 μS/m

طبيعة المواد العازلة

5.7 The Nature of Dielectric Materials p:127

إن العوازل تحتوي على شحنات (ثنائيات dipoles) مقيدة وليست طليقة ، بسبب القوى الذرية والجزيئية لكن من الممكن إزاحتها قليلا بواسطة مجالا خارجيا.

P_o : polarization (الاستقطاب): a dipole moment per unit volume (c/m^2)

ϵ : permittivity

ϵ_r : relative permittivity or **(dielectric constant)** ثابت العزل

The dielectric materials (solid, liquids, or gas) have ability to store electric energy by means of shift in the relative positive and negative charges aginst the normal molecular and atomic forces.

الجزيئات القطبية

Polar molecules: the molecules have permenent displacement existing between the center of gravity of the positive and negative charges. and each pair of charges acts as a dipole.

في الجزيئات القطبية، توجد ازاحة دائمية بين مركزي الشحنتين (الموجبة والسالبة) مكونة ثنائي كهربائي ثم عند تسليط مجال كهربائي خارجي سوف يعمل على ازاحة اضافية بين مركزي الشحنتين.

الجزيئات غير القطبية

Non polar malecules: the molecules that does not have a dipoles until after a field is applid.

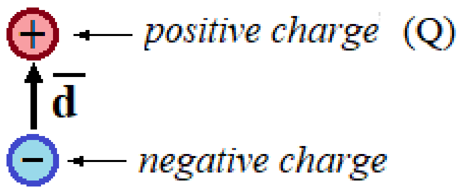
في الجزيئات غير القطبية لا توجد ازاحة بين مركزي الشحنات فلا توجد ثنائيات اصلا ولكن عند تسليط مجالا خارجيا فانه سيعمل على ازاحة معينة بين مركزي الشحنتين ، مكونة ثنائيات كهربائية

$$\overline{P} = Q\,\overline{d} \quad \text{(c.m)}$$

$\overline{P}$: dipole moment **(عزم الثنائي)** (c.m)

Q: the positive one of the two dipole charges

$\overline{d}$: the vector from -Q to +Q



dipole

If there are n dipole per unit volume:

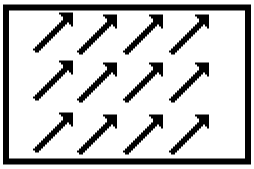
The total dipole moment is:

$$\boldsymbol{P}_{total} = \sum_{i=1}^{n\Delta v} P_i$$

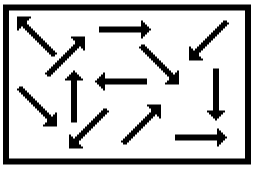
Polarization (الاستقطاب) $\boldsymbol{P}_\circ = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} \boldsymbol{P}_i$

$$\boxed{\boldsymbol{D} = \epsilon_\circ \boldsymbol{E} + P_\circ}$$

$$\boxed{\nabla \cdot \boldsymbol{D} = \rho_v}$$



$\mathbf{P}_{total}$ =(high)



$\mathbf{P}_{total} = 0$

$\boxed{P_\circ = X_e \, \epsilon_\circ \, E}$ (c/m²)

Xe: (chi) electric susceptibility of the material.

$$\boldsymbol{D} = \epsilon_\circ \epsilon_r \boldsymbol{E} = \epsilon \, \mathbf{E}$$

$$\boxed{\epsilon_r = X_e + 1}$$

ϵ: Permitivity $\boxed{\epsilon = \epsilon_\circ \epsilon_r}$

ϵ_r: The dielectric constants are given for some representative materials in Appendix C in the text book (hayt).

Example (1): A solid conductor has a surface described by: X+Y=3 m, and extends toward the origin. At the surface, the electric field intensity is 0.35 V/m. Express **E** and **D** at the surface and find the surface charge density?

Solution: The electric field and flux density is normal to the surface at the conductor surface.

To find a unit normal vector **aN**:

1- S = X+Y – 3 = 0

Normal vector: $\nabla \boldsymbol{S} = \frac{\partial s}{\partial x}\boldsymbol{ax} + \frac{\partial s}{\partial y}\boldsymbol{ay} + \frac{\partial s}{\partial z}\boldsymbol{az}$

$$\nabla \boldsymbol{S} = \boldsymbol{ax} + \boldsymbol{ay} = \boldsymbol{N}$$

$\boldsymbol{aN} = \frac{\boldsymbol{ax}+\boldsymbol{ay}}{\sqrt{2}}$ معادلة متجه الوحدة العمودي على السطح:

$$\boldsymbol{E_N} = 0.35\,\boldsymbol{aN} = \mathbf{0.35}\left[\frac{\boldsymbol{ax} + \boldsymbol{ay}}{\sqrt{\mathbf{2}}}\right] = \mathbf{0.2475}(\boldsymbol{ax}+\boldsymbol{ay})\ \boldsymbol{V/m}$$

$$\mathbf{D}_N = \epsilon_\circ\,\mathbf{E} = 8.854 \times 10^{-12} * 0.2475(\boldsymbol{ax}+\boldsymbol{ay})$$

$$\mathbf{D}_N = 2.19 \times 10^{-12}(\mathbf{ax}+\mathbf{ay})\ \ c/m^2$$

$$\rho_s = |\mathbf{D}_N| = (2.19 \times 10^{-12})\sqrt{2} = 3.0989 \times 10^{-12}\ \ C/m^2$$

Example (2): Two point charges in dielectric medium, where ϵr=5.2, interact with a surface of 8.6 mN what force could be expected if the charges were in free space.

Solution:

$$|F_1| = \frac{Q_1\,Q_2}{2\pi\epsilon_\circ r^2}$$ in free space

$$|F_2| = \frac{Q_1\,Q_2}{2\pi\epsilon r^2} = 8.6\times 10^{-3}N$$

$$\frac{|F1|}{|F2|} = \frac{\frac{Q_1Q_2}{4\pi\ \epsilon_\circ\ r^2}}{\frac{Q_1Q_2}{4\pi\ \epsilon\quad r^2}} = \frac{\frac{1}{\epsilon_\circ}}{\frac{1}{\epsilon_\circ\epsilon_r}} = \frac{\epsilon_\circ\epsilon_r}{\epsilon_\circ}$$

$$\frac{F1}{F2} = \epsilon_r \rightarrow \quad F1 = F2\epsilon_r = 8.6\times 10^{-3} * 5.2$$

|F1| = 44.72 mN

D5.8. A slab of dielectric material has a relative dielectric constant of 3.8 and contains a uniform electric flux density of 8 nC/m². If the material is lossless, find: (a) **E**; (b) **P**; (c) the average number of dipoles per cubic meter if the average dipole moment is 10^{-29} C· m.

Ans. 238 V/m;   5.89 nC/m²; 5.89×10^{20} m^{-3}

Solution"

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_\circ \epsilon_r} \rightarrow \ |\mathbf{E}| = \frac{|\mathbf{D}|}{\epsilon_\circ \epsilon_r}$$

$$\rightarrow \ |\mathbf{E}| = \frac{8 \times 10^{-9}}{\frac{10^{-9}}{36\pi} * 3.8} = 238.1 \ \ V/m$$

$$\boxed{\boldsymbol{D} = \epsilon_\circ \boldsymbol{E} + P_\circ} \rightarrow \ P_\circ = |\boldsymbol{D}| - (\epsilon_\circ |\boldsymbol{E}|)$$

$$\rightarrow \ P_\circ = 8 \times 10^{-9} - \ (8.854 \times 10^{-12} * \ 238.1)$$

$$= 5.8918 \times 10^{-9} \ c/m^2 \ \ or \ \ 5.892 \ nc/m^2$$

Average number of dipole/m³ = $\dfrac{P_\circ}{Average\ dipole\ moment}$

Average number of dipole/m^3 $= \dfrac{5.892 \times 10^{-9}}{10^{-29}}$

Average number of dipole/m³ = $5.892 \times 10^{20} \ m^{-3}$

$$free\ charged\ enclosed: Q \ = \oint_s \mathbf{D}\,.\mathbf{ds} \ldots\ldots\ldots(26)$$

Example: 5.4 **P: 132**

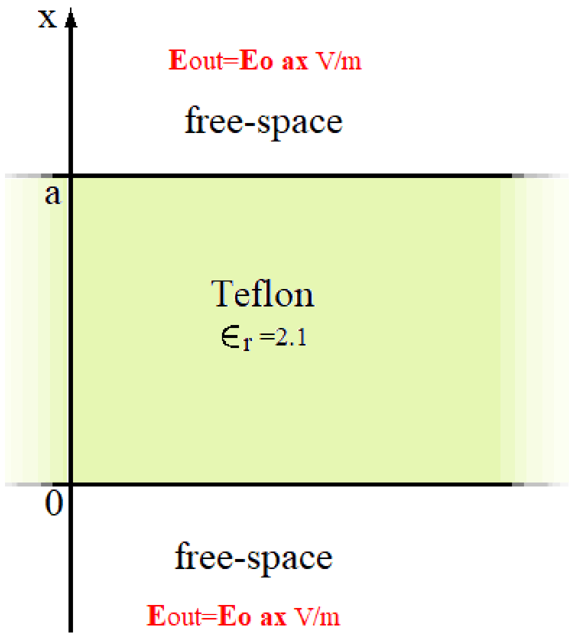
We locate a slab of Teflon in the region $0 \le X \le a$ and assume free-space

where $X < 0$ and $X > a$. Outside the Teflon there is a uniform field **Eout=Eo ax** V/m.

We seek values for **D, E** and **P** every where.

Note: *The dielectric constant of the Teflon is 2.1*

Solution:



$$\epsilon_r = X_e + 1 \rightarrow X_e = \epsilon_r - 1$$

$$= 2.1 - 1$$

$$X_e = 1.1 \; (electric\ susceptibility)$$

Outside: $\mathbf{D}_{out} = \epsilon_\circ \; \mathbf{E}_\circ \; \boldsymbol{ax}$

$$\mathbf{P}_{out} = 0 \; (no\ dielectric\ \boldsymbol{material})$$

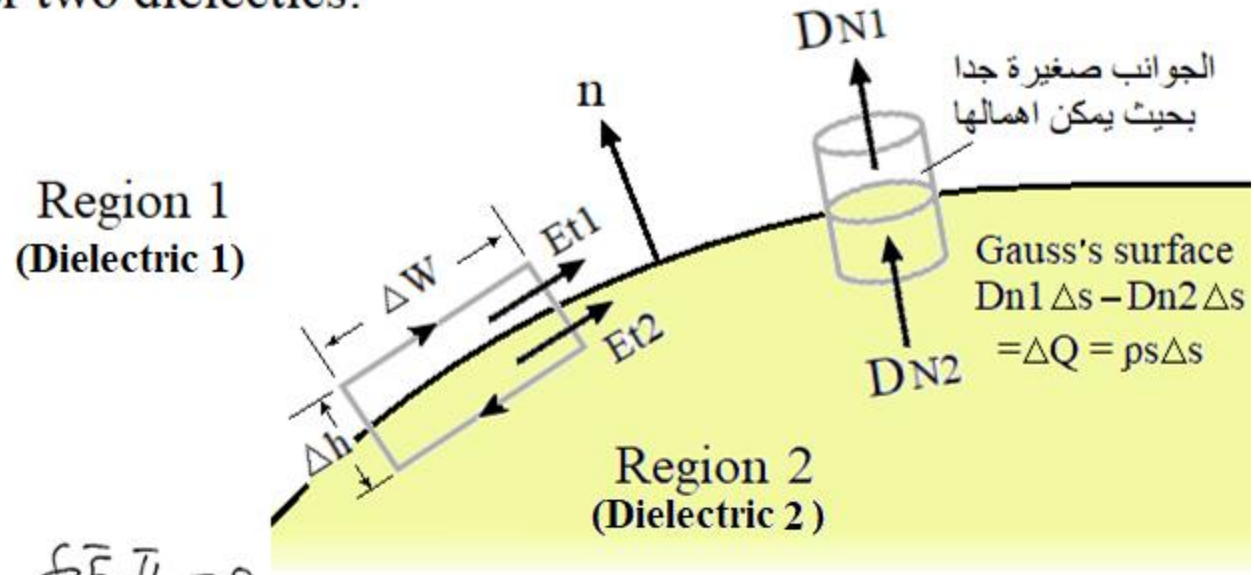
$$\mathbf{D}_{in} = \epsilon_\circ \epsilon_r \; \mathbf{E}_{in} = \mathbf{8.854 \times 10^{-12} * 2.1} * \boldsymbol{E_{in}} \; (0 \le X \le a)$$

$$\mathbf{P}_{in} = X_e \, \epsilon_\circ \, \mathbf{E}_{in} = 1.1 * 8.854 \times 10^{-12} \, \mathbf{E}_{in} \; (0 \le X \le a)$$

Fields crossing over the boundary from the known fields external to the dielectric to the unknown ones, to detetmine this we need a boundary condition, the subject fo the next section:

Boundary Conditions for Perfect Dielectric Materials

* For two dielectics:



$$\oint \bar{E} \cdot \bar{dL} = 0$$

around the small closed path (on the left):

$$E_{tan1}\,\Delta W - E_{tan2}\,\Delta W = 0 \qquad (\Delta h \overset{\sim}{\to} 0)$$

$$\boxed{E_{tan1} = E_{tan2}} \quad \text{----------} \quad (32)$$

$$\frac{D_{tan1}}{\epsilon_1} = E_{tan1} = E_{tan2} = \frac{D_{tan2}}{\epsilon_2}$$

$$\boxed{D_{N1} - D_{N2} = \rho_{s1}} \quad \text{----------} \quad (34)$$

assume ρ_s = Zero on the interface and : (for special case)

$$\boxed{D_{N1} = D_{N2}} \quad \text{----------} \quad (35)$$

$$\epsilon_1 E_{N1} = \epsilon_2 E_{N2} \quad \text{----------} \quad (36)$$

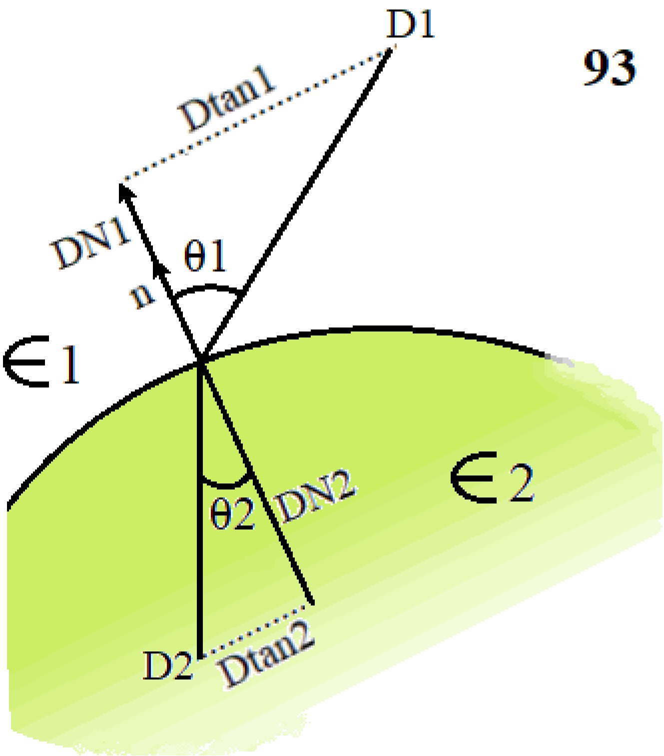
المعادلة العامة ل (32)

$$(\bar{D}_1 - \bar{D}_2) \cdot \bar{n} = \rho_s \quad \text{----------} \quad (37)$$

$$(\bar{E}_1 - \bar{E}_2) \times \bar{n} = 0 \quad \text{----------} \quad (38)$$

(normal component by dot $\bar{n}$; tangent component by $\times \bar{n}$)

المعادلة العامة ل(34) (استخدمنا سابقا: عند الموصلات (EXn=0



asume $\in 1 > \in 2 \longrightarrow \theta 1 > \theta 2$

Et1 = Et2

$$\frac{Dt1}{\in 1} = \frac{Dt2}{\in 2} \longrightarrow \frac{Dt1}{Dt2} = \frac{\in 1}{\in 2}$$

$$DN1 = D1 \cos \theta 1 = D2 \cos \theta 2 = DN2 \text{ ------(39)}$$

$$\frac{Dt1}{Dt2} = \frac{D1 \sin \theta 1}{D2 \sin \theta 2} = \frac{\in 1}{\in 2}$$

$$\in 2 \; D1 \sin \theta 1 = \in 1 \; D2 \sin \theta 2 \text{ ....(40)}$$

and the division of this eq. by (39) gives:

$$\frac{\tan \theta 1}{\tan \theta 2} = \frac{\in 1}{\in 2}$$

The direction of E on each side of the boundary is identical with the direction of D because D = ∈ E .

*** from eq. (39) and (40):**

$$D2 = D1 \sqrt{\cos^2 \theta_1 + \left(\frac{\in 2}{\in 1}\right)^2 \sin^2 \theta_1} \text{ ....(42)}$$

$$E2 = E1 \sqrt{\sin^2 \theta_1 + \left(\frac{\in 1}{\in 2}\right)^2 \cos^2 \theta_1} \text{ ....(43)}$$

Example 1: The surface X=0 separate two prfect dielectrics.

For X > 0 let ϵr1=3 , while ϵr2 =5 where X < 0 .

If **E1**= 80 **ax** – 60 **ay** – 30 **az** V/m , find:

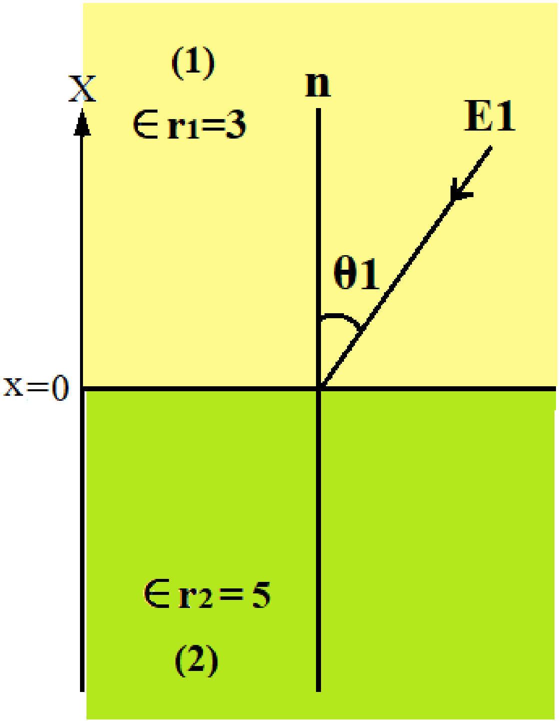
a) EN1 , b) Et1 , c) |**E1**| , d) θ1 , e) DN2 , f) Dt2 , g) **D2** , h) **P2** , i) θ2.

Solution: **E1**= 80 **ax** – 60 **ay** – 30 **az** V/m

a) **EN1** = 80 **ax** V/m

b) **Et1 = -**60 **ay** – 30 **az** V/m

c) $|E1| = \sqrt{80^2 + 60^2 + 30^2} = 104.4\ V/m$



d) $\cos\theta 1 = \frac{|EN1|}{|E1|} = \frac{80}{104.4} \rightarrow \theta 1 = Cos^{-1}\left(\frac{80}{104.4}\right) = 40^{\circ}$

e) **DN2** = **DN1** (for ρs=0) → **DN1** = $\epsilon_{\circ}\,\epsilon_r$ **EN1**

DN1 = $8.854 \times 10^{-12} * 3 * 80$ **ax** = 2.124 **ax** nc/m²

∴ **DN2** = 2.124 **ax** nc/m^2

f) **Dt2** = ? $\frac{\mathbf{Dt}1}{\in 1} = \frac{\mathbf{Dt}2}{\in 2} \rightarrow \mathbf{Dt}2 = \mathbf{Dt}1 \frac{\in 2}{\in 1}$

$\mathbf{Dt}2 = \in 1\ \mathbf{Et}1 \frac{\in 2}{\in 1} \rightarrow \mathbf{Dt}2 == 60\mathbf{ay} - 30\ \mathbf{az} * \frac{10^{-9}}{36\pi} * 5$

Dt2 = **-**2.66 **ay** – 1.33 **az** nc/m²

g) **D2 = DN2 + DT2 =** 2.12 **ax** – 2.66 **ay** – 1.33 **az** nc/m²

h) $\mathbf{P_2} = Xe2\ \in_\circ\ \mathbf{E2}\ ,\ Xe = \in_{r2} - 1 = 5 - 1 = 4$

$$P_2 = 4 * 8.854 * 10^{-12} * \frac{2.12\ \mathbf{ax} - 2.66\ \mathbf{ay} - 1.33\ \mathbf{az}}{8.854 * 10^{-12} * 5}$$

$$P_2 = 0.8\ (2.12\ \mathbf{ax} - 2.66\ \mathbf{ay} - 1.33\ \mathbf{az})\ c/m^2$$

i) θ2 = ? $\frac{tan\ \theta 1}{tan\ \theta 2} = \frac{\epsilon_1}{\epsilon_2}$

$$\frac{tan\ 40}{tan\ \theta 2} = \frac{\epsilon_0\ \in_{r1}}{\epsilon_0\ \in_{r2}} \rightarrow tan\ \theta 2 = tan(40) - \frac{5}{3} = 1.4$$

$$\theta 2 = tan^{-1} 1.4 = 54.433°$$

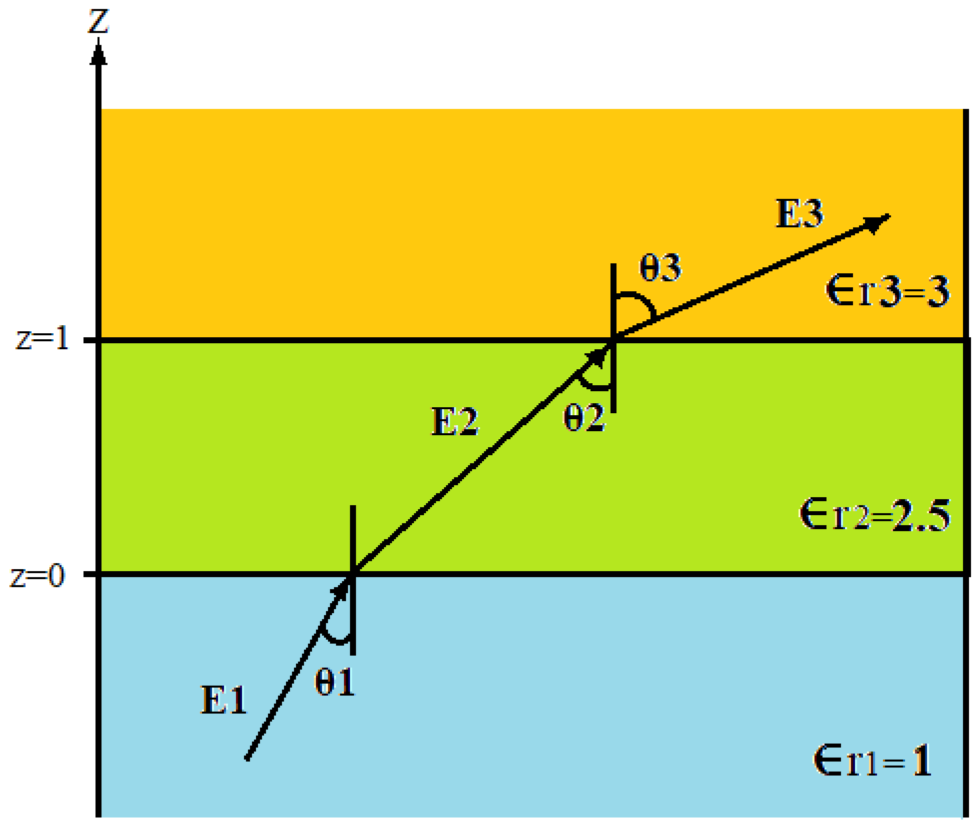
Example 2: Region 1 (Z < 0 m) is free space where **D**=5**ay**+7**az** c/m²

Region 2 (0 < z < 1m) has ϵr = 2.5

Region 3 (z > 1m) has ϵr = 3

Find: **E2**, **P2** and θ3.

Solution:



D1 = 5 **ay +** 7 **az** c/m²

DN1=D1 cos θ1

$\text{Cos}\theta 1 = \frac{7}{\sqrt{5^2+7^2}} \rightarrow \theta 1 = 35.53^{\circ}$ **D1**=5**ay**+7**az** c/m²

DN1 = **DN2** = 7 **az**

Et1 = Et2 $\rightarrow \frac{Dt1}{\epsilon_1} = \frac{Dt2}{\epsilon_2} \rightarrow Dt2 = \frac{\epsilon_2\, Dt1}{\epsilon_1}$

$$\mathbf{Dt2} = \frac{\in_{r2}\ \mathbf{Dt1}}{\in_{r1}} = \frac{\mathbf{2.5}}{\mathbf{1}}\ \mathbf{5\ ay} = \mathbf{12.5\ ay}\quad \mathbf{c}/\boldsymbol{m}^{\mathbf{2}}$$

D2 = DN2 + Dt2 = 12.5 **ay** + 7 **az** $\mathbf{c}/\boldsymbol{m}^{\mathbf{2}}$

$$\mathbf{E2} = \frac{\mathbf{D2}}{\in_2} = \frac{12.5\ \mathbf{ay} + 7\ \mathbf{az}}{2.5 \in_\circ} = \frac{1}{\in_\circ}\left(5\ \mathbf{ay} + \frac{7}{2.5}\mathbf{az}\right)$$

DN2 = D2 cos θ2

$$\theta 2 = cos^{-1}\frac{7}{\sqrt{12.5^2 + 7^2}} = 60.7^\circ$$

$$\frac{tan\ \theta 2}{tan\ \theta 3} = \frac{\epsilon_2}{\epsilon_3} \rightarrow tan\ \theta 3 = \frac{\epsilon_{r3}}{\epsilon_{r2}}\ tan\ \theta 2$$

$$tan\ \theta 3 = \frac{3}{2.5}\ tan\ 60.7$$

$$\boldsymbol{\theta 3} = \mathbf{64.93}^\circ$$